EXACT ANALYTICAL SOLUTIONS FOR ASYMMETRIC SURGING AND SURF-RIDING

K.J. Spyrou

Department of Naval Architecture and Marine Engineering
National Technical University of Athens
9 Iroon Polytechniou, Zographou, Athens 15773, Greece

Extended abstract

It is well-known that the surging behaviour of a ship in following waves of substantial steepness can become strongly nonlinear for a certain range of vessel speeds (normally for Froude numbers higher than 0.3). This nonlinear behaviour is manifested with a gradual change of the ship's response towards an asymmetric pattern of surging even if the considered wave is of a simple sinusoidal form. Typically, the ship spends more time on the crests and less on the troughs, a tendency which becomes more pronounced as the speed is increased further. Furthermore, unusual types of behaviour with a stationary nature and in competition with the periodic pattern arise, featuring a forced motion at a speed equal to the wave celerity and with the ship "locked" between two consecutive wave crests. These phenomena have already been studied on the basis of numerical models for a following as well as for a quartering sea environment and specific explanations of their dynamics have been produced[1]. A typical phase-plane plot of nonlinear surging is shown in Fig.1.

We have recently endeavoured to develop a purely analytical description for the surging behaviour, even for the strongly nonlinear range[2]. This would be very useful for design where closed-form expressions are always preferred. It would benefit also advanced investigations on phenomena such as broaching and loss of transverse stability.

The differential equation of surging motion has a strong nonlinearity in the stiffness term since the wave force is a sinusoidal function of position. There is also a weak nonlinearity in the damping (=difference between resistance and thrust). A general form for the equation of surge on a sinusoidal wave is,

$$(m - X_{\dot{u}})\dot{u} + [R(u,c) - T(u,c,n)] + f\sin(kx) = 0 \qquad (1)$$

The velocity relatively to the wave is $\dot{x} = c - u$. With substitution of suitable polynomial expressions for the thrust and the resistance, eq. (1) becomes,

$$(m - X_{\dot{u}})\ddot{x} + \left\{\left[3r_3 c^2 + 2(r_2 - \tau_0)c + r_1\right] - \tau_1 n\right\}\dot{x} + \left[3r_3 c + (r_2 - \tau_0)\right]\dot{x}^2 + r_3\dot{x}^3 + f\sin(kx) = \\ = \underbrace{\left(\tau_0 c^2 + \tau_1 cn + \tau_2 n^2\right)}_{T(c;n)} - \underbrace{\left\{r_1 c + r_2 c^2 + r_3 c^3\right\}}_{R(c)} \qquad (2)$$

a) *Surf-riding*:

This calculation is straightforward: If $-1 \le \frac{T(c;n) - R(c)}{f} \le 1$ then $-1 \le \sin kx \le 1$. Stationary solutions become possible (surf-riding), located at,

$$x = \frac{2\nu\pi}{k} + \frac{1}{k}\arcsin\left[\frac{T(c;n) - R(c)}{f}\right], \quad x = \frac{(2\nu+1)\pi}{k} - \frac{1}{k}\arcsin\left[\frac{T(c;n) - R(c)}{f}\right] \qquad (3)$$

b) *Asymmetric surging*:

Equation (1) is brought into the following form (4), after substituting the damping terms with an equivalent quadratic on the basis of a least-square fit,

$$(m - X_{\dot{u}})\ddot{x} + \gamma(c;n)\dot{x}|\dot{x}| + f\sin(kx) = T(c;n) - R(c) \tag{4}$$

For an overtaking wave (4) leads to the following expression for the orbits of the phase plane $(x,\ \dot{x})$,

$$\frac{dx}{dt} = \dot{x} = -\frac{1}{k}\sqrt{c_2 q e^{2pkx} + \frac{2q(\cos kx + 2p\sin kx)}{(1+4p^2)} - \frac{r}{p}} \tag{5}$$

The term $c_2 q e^{2pkx}$ represents the transient part of the solution and it vanishes gradually since $x \to -\infty$ (the ship is trailing behind the waves). Therefore the expression for the steady periodic motion is,

$$\frac{dx}{dt} = \dot{x} = -\frac{1}{k}\sqrt{\frac{2q(\cos kx + 2p\sin kx)}{(1+4p^2)} - \frac{r}{p}} \tag{6}$$

With suitable transformations it can be shown that (6) can be solved explicitly for t,

$$t = -\frac{\sqrt{m}}{a}\mathrm{F}\left(\frac{kx-\theta}{2},\ m\right) \tag{7}$$

where F is the elliptic integral of the first kind, $\mathrm{F} = \int_0^{\varphi} \frac{1}{\sqrt{1 - m\cos^2\vartheta}}\,d\vartheta$ with modulus $\sqrt{m}$. In Fig 2 is shown the relation between time and position which shows clearly the distortion from the linear pattern. With inversion of (7) we obtain further the following expression for x in terms of t (see also Fig. 3),

$$\cos(kx - \vartheta) = 1 - 2\,\mathrm{sn}^2\left(-\frac{a}{\sqrt{m}}t,\ \sqrt{m}\right) \tag{8}$$

When the ship operates away from the strongly nonlinear regime, the *rhs* of (8) tends to obtain the cyclic form $\cos(\omega_e t)$

c) *Condition for heteroclinic connection*

The heteroclinic connection leads to the disappearance of the periodic motion. This phenomenon is linked with broaching and it happens as soon as the unstable stationary solution near the crest falls on the periodic orbit.

Unstable stationary point:
$$\dot{x} = 0, \quad x = \frac{(2\nu - 1)\pi}{k} - \frac{1}{k}\sin^{-1}\frac{T(c;n) - R(c)}{f} \tag{9}$$

The steady periodic orbit is given by eq. (6). Substitution of (9) into (6) yields the following expression of critical amplitude for the surge wave force,

$$f_{crit} = \frac{R(c;n) - T(n)}{2\gamma}\sqrt{k^2 (m - X_{\dot{u}})^2 - 4\gamma^2} \tag{10}$$

References

[1] K.J. Spyrou (1996) Dynamic instability in quartering seas: The behaviour of a ship during broaching, *Journal of Ship Research*, SNAME, USA, Vol. 40, No. 1, pp. 46-59.

[2] K.J. Spyrou (2000) On the parametric rolling of ships in a following sea under simultaneous nonlinear periodic surging. *Philosophical Transactions of the Royal Society of London*, A 358, pp. 1813-1834.

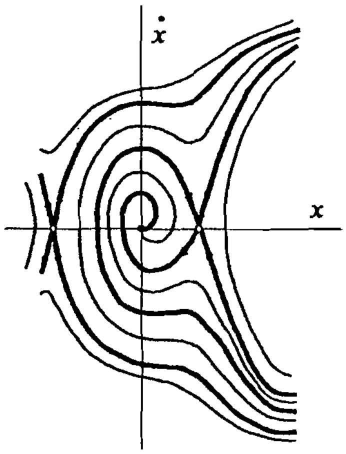

Fig. 1: Typical phase-plane orbits in the range of surf-riding.

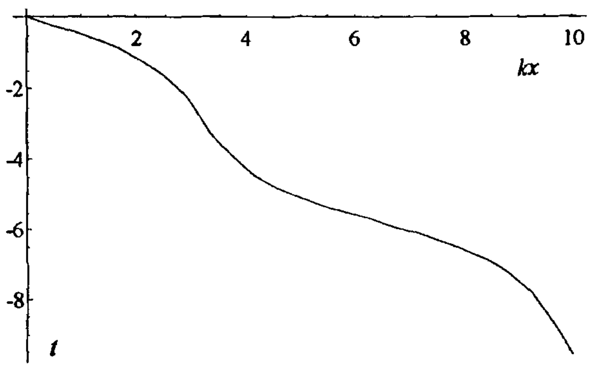

Fig. 2: The relation between time and position is distorted from the linear one in the range of asymmetric surging.

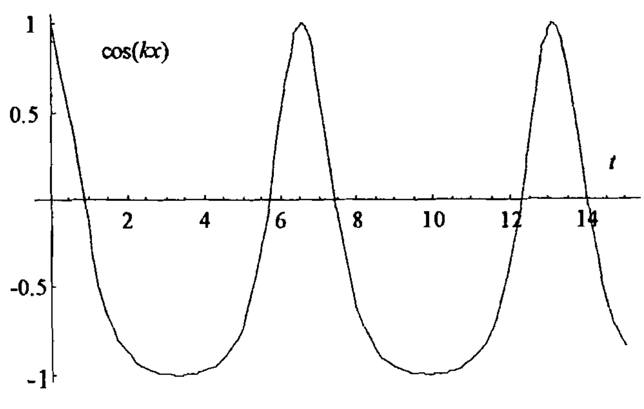

Fig. 3: Surge as described by eq. (8).