Proceedings of the 6th Osaka Colloquium on Seakeeping and Stability of Ships (OC 2008), 26 - 29 March 2008, Osaka, Japan

NOVEL MODELLING OF SHIP ROLLING BASED ON FRACTIONAL CALCULUS

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ABSTRACT

A step towards a novel formulation of ship motions modeling that is based on the so called "fractional differential equations" is presented. An interesting feature of such an approach is that it can be host of hydrodynamic terms that lie between damping and added inertia, thus removing, at least in principle, the necessity of their separate representation. Notably, memory is an intrinsic feature of differential equations that include fractional derivatives. Thus a thought-provoking link is provided with the well-known "convolution integral" formulation. Preliminary implementation of this idea is attempted targeting linear and uncoupled ship rolling.

KEY WORDS: Ship, Roll, Fractional Calculus, Memory, Damping, Added Inertia, Convolution.

INTRODUCTION

Froude's (1861) celebrated analogy of a rolling ship to a rotational oscillator and its subsequent extension to all ship motions by Kriloff (1898) were based on distinctive terms for inertia, damping and stiffness. Later awareness of the frequency dependence of hydrodynamic inertia and damping produced the well-known integro-differential structure of the "time-domain" ship motion equations (Haskind 1953; Golovato 1959; Tick 1959; Cummins 1962; Ogilvie 1964 and others). That was in fact consistent with a Volterra series expansion of hydrodynamic reaction (Bishop et al. 1973). Yet the oscillator concept was not fundamentally challenged despite the fact that the frequency-dependent added moment of inertia and damping coefficients are derived from the sine and cosine Fourier transforms of the same kernel function. We have wondered whether a formulation that would not stipulate separation between hydrodynamic inertia and "potential" damping on the basis of the integer order derivative of displacement/rotation could be viable, through some consistent, yet perhaps not so ordinary, mathematics.

One alternative perspective that appears promising and is explored in the current note accrues from the notion of *fractional calculus* that can accommodate non-integer order derivatives (see for example Gorenflo & Mainardi 1997). An interesting possibility within this different mathematical framework is that, hydrodynamic terms that lie between damping and added inertia can appear legitimately in the motion equations. Imagine for example a term in the mathematical model that is based on an $1+\alpha$ order derivative of the roll angle, where α is some real number between 1 and 2. Mathematical techniques that can handle such a situation are currently available. It is remarkable however that, a conventional model is still capable to replicate the behaviour of a fractional one by means of use of convolution integrals. In other words, any results achieved are not really beyond the realm of current standard tools of analysis. However the fact that convolution integrals are involved proves that "memory" is an intrinsic feature of differential equations that include derivatives of non-integer order and, as a matter of fact, of a ship motions formulation that employs such entities.

The prospects of an approach that follows this principle will be evaluated to some extent, for ship motion applications. The investigation is focused on some simple scenarios: decaying roll motion, (direct) roll resonance; and parametric resonance.

ESSENTIALS OF FRACTIONAL CALCULUS

The idea of calculating derivatives and integrals of non-integer order is perhaps received by the traditionally educated engineer with reservation. However, a renowned letter exchange between Leibnitz and L' Hopital in 1695 proves that the interest for the subject runs back to the era when calculus was 'born'. In fact, mathematicians of the likes of Euler, Lagrange, Laplace, Riemann, Fourier and Liouville have dealt with it. Glamorous past notwithstanding, one feels however that the field is still at the stage of infancy. Geometrical interpretation of the fractional derivative is still obscure (for an attempt see Podlubny 2002). Furthermore, and despite current interest, one is still bemused by several competing mathematical conventions on how fractional order differentiation and integration should be performed (Appendix I).

Irrespectively of the adopted definition, the concept enables one to write and solve differential equations with terms of fractional order under the following general form:

$$c_n D^{\alpha_n} x(t) + c_{n-1} D^{\alpha_{n-1}} x(t) + \cdots + c_1 D^{\alpha_1} x(t) + c_0 D^{\alpha_0} x(t) = F(t) \qquad (1)$$

where a_k ($k = 0, 1, \ldots, n$) are positive real numbers and the terms $D^{a_k} x(t)$ stand for the fractional derivatives that appear in this equation. The dynamical behaviour 'produced' by nonlinear fractional differential equations has already been the focus of attention by a number of researchers (see for example Achar et al 2001 & 2002). Moreover, interesting engineering applications have started to emerge: in one case, use of a fractional term (of order between 0 and 1) was proposed for representing the damping of a suspension bridge (Rossikhin & Shitikova 1998). In another case, the modelling of radiation damping in unbounded domains by fractional calculus has been contemplated (Trinks & Ruge 2003). A lot of interest is noted also on viscoelastic systems and on applications in control engineering. Naturally one wonders if such a viewpoint could find a place in the analysis of ship motions too.

ROLL-LIKE DYNAMICS FROM THE PERSPECTIVE OF FRACTIONAL DIFFERENTIAL EQUATIONS

A rudimentary linear model of ship rolling with one degree of freedom has been considered in the first instance. Added moment of inertia and potential damping were accounted through a single fractional derivative of the roll angle φ. More appropriate would have been perhaps if several fractional order terms had been introduced, according to specific energy dissipation mechanisms. To simplify matters let us neglect any effect of viscous origin.

$$I\ddot{\varphi} + B\varphi^{(1+\alpha)} + mg\overline{GM}\,\varphi = f(t) \qquad (2)$$

As observed, no hydrodynamic part has been added to the mass moment of inertia I. To allow the fractional derivative to act as bridge between added inertia and damping, the value of the fractional order has been confined in the range $0 \le \alpha \le 1$. Through discretization of equation (2) one can obtain a numerical solution based on an iteration scheme like the following [for the derivation consult for example Podlubny (1999)]:

$$\varphi_i = \frac{f_i - mg\,GM\,\varphi_{i-1} + I\,h^{-2}\left(2\varphi_{i-1} - \varphi_{i-2}\right) - B\,h^{-(a+1)} \sum_{j=1}^{i} w_j^{(a+1)} \varphi_{m-j}}{I\,h^{-2} + B\,h^{-(a+1)}} \qquad (3)$$

$$i = 2, 3 \ldots$$

In the above, h is the time step, $f_i = f(ih)$ and $\varphi_i = \varphi(ih)$. One problem is that homogeneous initial conditions are required for applying the above scheme. This can often be overcome however, through suitable transformation of the time dependent variable. The iteration scheme for identifying the terms $w_j^{(a)}$ whose presence is noted in (3) (they are in fact defined by the expression $w_j^{(a)} = (-1)^j \binom{a}{j}, \quad j = 0, 1, 2\ldots$), is given below:

$$w_0^{(a)} = 1; \quad w_j^{(a)} = \left(1 - \frac{a+1}{j}\right) w_{j-1}^{(a)}, \quad j = 1, 2, 3\ldots, \quad a \in [0,1] \qquad (4)$$

Observations based on freely decaying motion

Let us firstly examine the influence of the fractional order on the period of a decaying oscillation and also on the drop of amplitude per cycle. A reference ship was assumed with $\omega_0 = 0.431$ rad/s and a damping ratio $\zeta = 0.0356$ (no explicit hydrodynamic inertia term was involved in these calculations). The reference ship had $GM = 2.08$ m, $m = 1.631 \times 10^8$ kg, $I = 1.791 \times 10^9$ kg m^2, $B = 5.5 \times 10^7$ N m s. Examples of the numerically obtained time-histories of decaying roll motion, initiated from a 10^0 angle, are shown in Fig. 1. The parameter in these graphs is the fractional order a (it is reminded that the order of the derivative is in fact 1+ a), that was thus set to assume values between damping and inertia (0.1, 0.3 and 0.6). Attesting the validity of the numerical scheme, when a approached an integer value (0 or 1) the simulated response coincided with that obtained through conventional calculus.

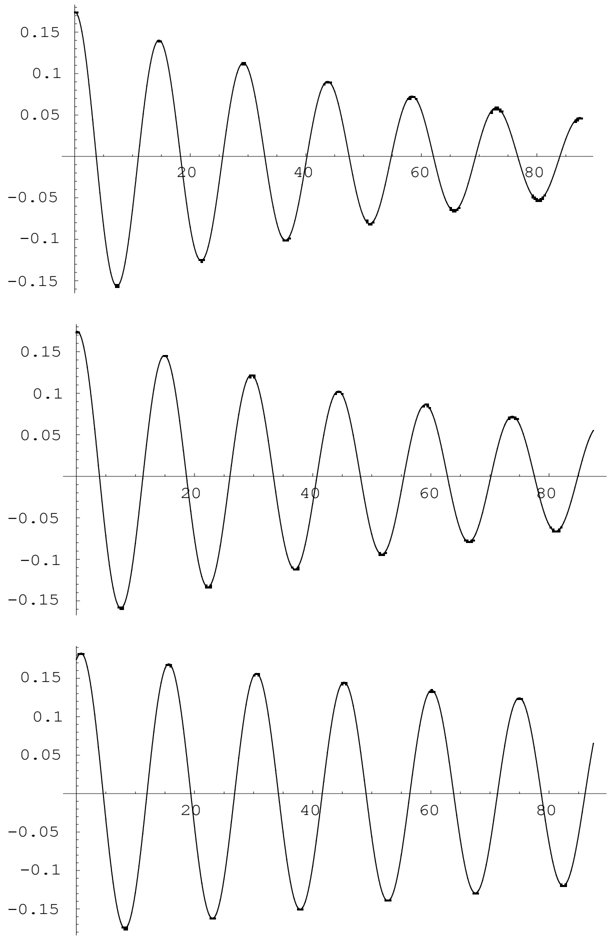

Fig. 1: Characteristic decays for fractional orders a that span the range between pure damping and pure inertia: a=0.1 (upper); a=0.3(middle); a=0.6 (lower).

The influence of the pair a, B on the period of oscillation can be probably deduced from the diagram like the one shown in Fig. 2: the effect on the period becomes noticeable when the fractional order exceeds a value about 1.6. Furthermore, for low values of the coefficient of the fractional term there is no substantial effect on the period, throughout the current range of values of the fractional order. But this was perhaps expected: unless the damping had been large in the first place, as the damping coefficient is gradually transformed into an inertial one, the process is incapable of producing any substantial increase of inertia. This in turn is reflected upon the way the natural period changes.

The effect upon the drop of roll amplitude per cycle could be examined more systematically by introducing an "apparent" damping ratio ζ for the fractional system. The law of reduction of this apparent ζ as the fractional order is raised should say a lot about how the fractional oscillator could be imitated by a standard one. Indeed one recalls that for a linear decay the following holds:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} \approx \frac{\delta}{2\pi} \quad \text{where} \quad \delta = \ln\frac{A_n}{A_{n+1}} \tag{5}$$

where A_n is the peak of the n^{th} cycle. Then for a given B the relationship between ζ and α can be identified on the basis of the decay curves. The result that was obtained for the specific set of values presented earlier is shown in Fig. 2.

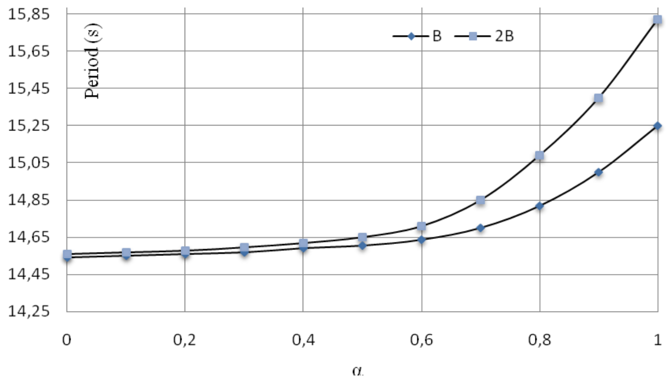

Fig. 2: Effect of *a* on the period of decaying oscillation, at two different values of the fractional coefficient.

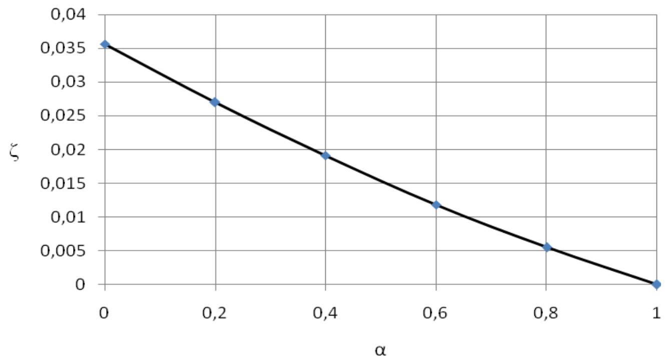

Fig. 3: The relationship between ζ and α is basically linear (the curve is based on approximate measurements of drop of amplitude per cycle).

We were thence inspired to examine whether the decaying motion simulated by a standard "integer-order" roll equation could be reasonably reproduced through optimal selection of the fractional order a and coefficient B. Let the roll equation that is to be reproduced to have the usual form:

$$(I+\delta I)\ddot{\varphi}(t) + B\ddot{\varphi}(t) + mg\,\overline{GM}\,\varphi(t) = 0 \tag{6}$$

Of course one may not disregard here that, in comparing the behaviour of an oscillator with memory against one without, the behaviour of an infinite dimensional system is judged against that of a low dimensional one. Convergence of the two could be claimed only in an approximate sense and basically when the memory is "short". On the other hand it is well-known that such an o.d.e. is often used in practice for modeling ship rolling. For the hydrodynamic moment of inertia we have assumed the value 1.718×10^8 kg m^2, based on some calculations.

Very good coincidence of decays has been obtained for at least one combination of α and B values. For the comparison shown in Fig. 4 the value of α was 1.50. It appears therefore that, by suitable selection of the "fractional" constants α and B, it is possible to achieve very good resemblance between the "fractional" and the conventional system behaviour; despite the fact that the first involves no customary terms for hydrodynamic inertia and damping. Currently the identification method is heuristic and it is unknown how this could accrue from system hydrodynamics on the basis of first principles.

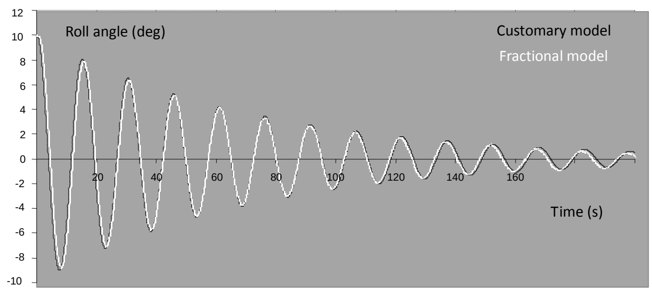

Fig. 4: Comparison of "integer" and "fractional" decays.

<u>*Roll-like oscillation with direct external excitation*</u>

As second step the resonant behaviour of the addressed category of fractional systems has been examined. The pair *a*, *B* played the role of varying parameters during this investigation. A roll model with external excitation was thus used, emulating the rotational behaviour of a ship that is excited by harmonic waves from abeam. The excitation term at the right-hand-side of equation (2) was calculated by the well-known expression:

$$f(t) = I\,Ak\,\omega^2\cos\omega t \tag{7}$$

The term *Ak* representing wave steepness was fixed at $1/20$ whereas the wave frequency ω was varied in the range around the roll natural frequency.

The resonance diagrams, obtained by running a large number of simulations and recording the steady sate amplitude; then repeating for the next value of α until completion, confirm that there is indeed a reduction of the "apparent" damping as the fractional order is raised (Fig. 5). One might recall here the following expression for the amplitude of the linear, lowly damped oscillator at resonance, which could be used for deducing combinations of $\zeta, \delta I$ (as usual δI stands for the hydrodynamic inertia) that produce equal roll amplitudes to those of the fractional system :

$$\varphi_0 = \frac{I}{I+\delta I}\frac{Ak}{2\zeta} \tag{8}$$

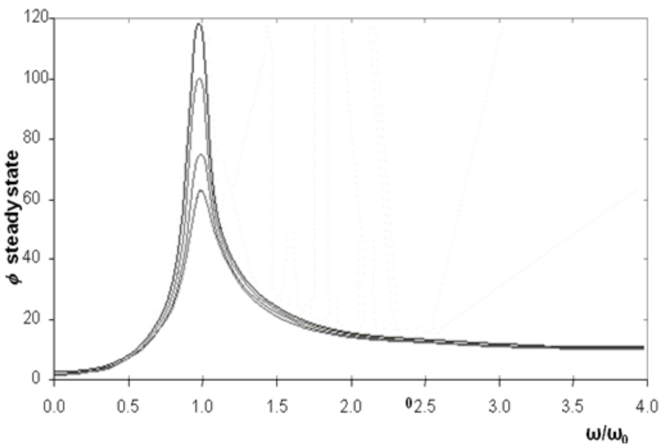

Fig. 5: Resonance diagrams of fractional system for *a* raised gradually from 0 (lowest curve) to 1.2, 1.4 and 1.5 (highest).

Response to parametric excitation

In the final investigation the fractional oscillator has been subjected to harmonic parametric excitation. The intention was to have a first shot on the effect invoked by the fractional order a upon the boundaries of the two first regions of parametric instability. The considered equation was thus:

$$\ddot{\varphi}(t) + 2\zeta\,\omega_0 D^{\alpha}\,\varphi(t) + \omega_0^2\left(1 - h\cos\omega_e t\right)\varphi(t) = 0 \tag{9}$$

As commonly, h is the amplitude of parametric excitation and ω_e is the frequency of encounter. The value of damping ratio for the reference"integer" system was assumed in this case $\zeta = 0.06$. From standard analysis of a damped Mathieu oscillator it is possible to infer that the critical parametric excitation amplitude for the first instability region should be encountered at about $h \approx 4\zeta = 0.24$. A characteristic parametric growth produced by the fractional oscillator, as compared to the growth of the conventional one, can be seen in Fig. 6, for h equal to 0.25 that is slightly higher than the critical value of the "integer" system.

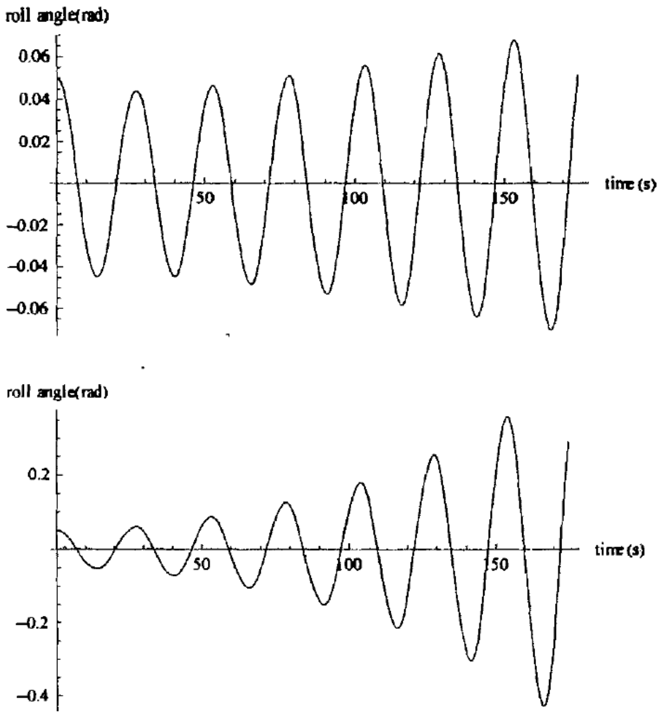

Fig. 6: Simulations of parametric growth for α=0 (upper) and α=0.6 (lower) at exact principal resonance. In the lower graph the growth is apparently faster, reminiscent of the presence of lower damping.

At this stage a large number of simulations was launched, scanning the ($4\omega_0^2/\omega_e^2$, h) plane with a small step in order to detect the combinations of ($4\omega_0^2/\omega_e^2$, h) parameters that produce oscillatory growth from some initial disturbance. The result can be seen in Fig. 7, for three different values of the fractional order α. The anticipated layout, featuring a gradual shifting upwards of the instability regions was indeed captured. A more focused study will be required however in order to comprehend what is the quantitative difference of the boundary lines of parametric instability of the fractional oscillator from those of an ordinary damped Mathieu system.

FINAL REMARKS

On the basis of the rather phenomenological approach applied here one might conclude that, fundamentally, a fractional oscillator is worthy of further investigation in relation to the modelling of ship rolling. Correspondence between the fractional and the ordinary "roil-like" oscillator was indicated by following a heuristic approach. Increase of the fractional order from pure damping to pure inertia is indeed reflected upon the response by a reduction of the apparent damping. At the same time the natural period is influenced by an 'added moment' like effect. It appears that there is scope for considering such an approach at a higher level of detail and for the modelling of ship motions in general.

The approach is more intuitive than conventional analysis. Nonetheless it is a challenge for the future how to link the fractional order with some fundamental concepts of ship hydrodynamics.

The revival of interest for fractional derivatives was probably sparked by their alluding connection with the fractal dimension of some nonlinear dynamical systems. Conferences and books were devoted to it but to no avail. But it seems that there are several much more direct benefits to be pursued for mainstream applications in fields of science and engineering.

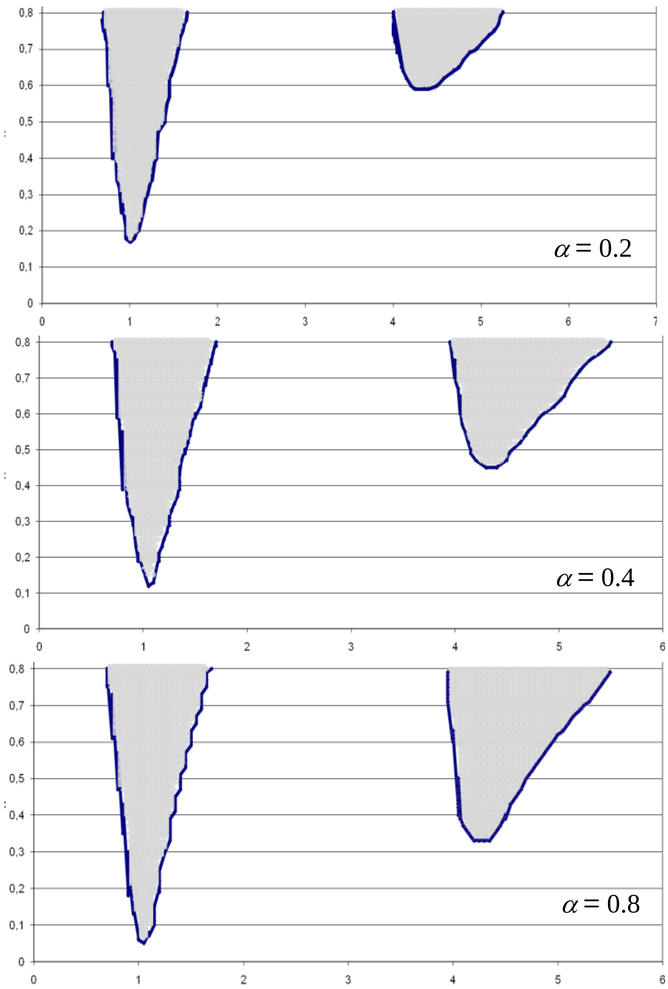

Fig. 7: Regions of principal and fundamental instability for three characteristic values of the fractional order.

REFERENCES

Achar, B.N.N,, Hanneken, J.W., Enck, T. & Clarke T. (2001) Dynamics of the fractional oscillator. *Physica A*, **297**, pp. 361-367.

Achar, B.N.N,, Hanneken, J.W. & Clarke T. (2002) Response characteristics of a fractional oscillator. *Physica A*, **309**, pp. 275-288.

Bishop, R.E.D., Burcher, R.K. & Price, W.G. 1973 The uses of functional analysis in ship dynamics. *Proceedings A*, The Royal Society, London, **332**, pp. 23-35.

Cummins, W.W. (1962) The impulse response function and ship motions. *Schiffstechnik*, **9**, 47, pp. 101-109.

Debnath, L. (2002) Fractional calculus fundamentals. Tutorial workshop 2: *Fractional calculus applications in automatic control and robotics*. 41st IEEE Conference on Decision and Control, Las Vegas,

Froude, W.C. (1861) On the rolling of ships, *Transactions of the Institution of Naval Architects*, **2**, pp. 180-229.

Golovato, P. (1959) A study of the transient pitching oscillations of a ship. *Journal of Ship Research*, **2**, 4, pp. 22-30.

Gorenflo, R. & Mainardi, F. (1997) Fractional calculus: integral and differential equations of fractional order. In *Fractals and Fractional Calculus in Continuum Mechanics*. Edited by A. Carpintieri & F. Mainardi, Springer (Wien, New York), ISBN 3-211-82913-X.

Haskind, M.D. (1953) *Oscillation of a ship on a calm sea*, Translation in SNAME Bulletin No 1-12.

Kriloff, A. (1898) A general theory of the oscillations of a ship in waves. *Transactions of the Institution of Naval Architects*, **40**, pp. 135-196.

Miller, K.S. & Ross, B (1993) *An introduction to the fractional calculus and fractional differential equations*, John Wiley & Sons, New York, ISBN 0471588849.

Ogilvie (1964) Recent progress toward the understanding and prediction of ship motions. *Proceedings*. 5th Symposium on Naval Hydrodynamics, Bergen. ACR-112, pp. 3-79.

Podlubny, I. (1999) *Fractional differential equations*. Volume 198 in Mathematics in Science and Engineering, Academic Press, ISBN 0-12-558840-2.

Podlubny, I. (2002) Geometric and physical interpretation of fractional integration and fractional differentiation. *Fractional Calculus & Applied Analysis*, **5**, 4, pp. 367-386.

Rossikhin, Y.A. & Shitikova, M.V. (1998) Application of fractional calculus for analysis of nonlinear vibrations of suspension bridges. *Journal of Engineering Mechanics*, **124**, 9, pp. 1029-1036.

Spanos, P.D. & Zeldin, B.A. (1997) Random vibration of systems with frequency-dependent parameters or fractional derivatives. *Journal of Engineering Mechanics*, **123**, 3, pp. 290-292.

Tick, L.J. (1959) Differential equations with frequency-dependent coefficients, *Journal of Ship Research*, **2**, 4, pp. 45-46.

Trinks, C.& Ruge, P. (2003) Fractional calculus applied to radiation damping, *Proceeding of Applied Mathematical Mechanics* (*PAMM*), 2, pp. 266–267.

APPENDIX I: PROPERTIES OF FRACTIONAL DERIVATIVES

There are currently several definitions of the fractional derivative. The most popular definitions together with their key properties are summarized below:

Riemann-Liouville fractional derivative

The Riemann-Liouville approach is easily grasped if one recalls the well-known Cauchy formula of n^{th} order integration:

$$J^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)\, d\tau, \quad t>0 \tag{I1}$$

where J^n is the operator of the integral.

This definition may be extended to any positive real order of integration because the right-hand-side is meaningful even for non-integer n (let's call it α to emphasise this generalisation):

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \ t>0 \tag{I2}$$

The fractional derivative can now be introduced by taking the standard, m integer-order derivative of the fractional integral:

$$D^\alpha f(t) = D^m J^{m-\alpha} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau\right]$$

$$m-1<\alpha\le m \tag{I3}$$

It is apparent that the fractional derivative works like a Volterra-type convolution integral, bringing-in memory and increasing to infinity the dimension of the system's state space. As pointed out by Spanos & Zeldin 1997 the fractional model may indeed be explicitly treated as a special case of one with frequency-dependent coefficients.

The integration might not start from zero but from some other instant $t=a_0$. For this reason one find in the literature also the so-called Riemann fractional derivative (Miller & Ross 1993):

$$D^\alpha_{a_0} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_{a_0}^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau\right] \tag{I4}$$

Also, in the so-called Liouville fractional derivative the left terminal is set at $-\infty$.

It general, in a composition of a Riemann-Liouville derivative with an integer (or fractional) order derivative, orders do not commute. More specifically:

$$\frac{d^n}{dt^n}\left[D^\alpha_{a_0} f(t)\right] = D^{n+\alpha}_{a_0} f(t) \tag{I5a}$$

$$D^\alpha_{a_0}\left[\frac{d^n}{dt^n} f(t)\right] = D^{\alpha+n}_{a_0} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a_0)(t-a_0)^{k-\alpha-n}}{\Gamma(1+k-\alpha-n)} \tag{I5b}$$

One observes that, in order to commute, function f and its n-1 derivatives should be zero at the lower terminal.

The Laplace transform of (I4) is (Debnath 2002):

$$L\left[D^\alpha_{a_0} f(t)\right] = s^\alpha \hat{f}(s) - \sum_{k=0}^{m-1} s^k D^{\alpha-k-1} f(0+) \tag{I6}$$

The Laplace transform requires the initial values of the fractional derivative at t=0. However, it is hard to imagine a physical meaning for terms like $D^\alpha f(0+)$. For this reason, and in order to facilitate the use of fractional calculus for the mathematical modeling of physical processes, Caputo proposed an alternative definition where the initial conditions can be interpreted in the customary way (see the specific paragraph later).

It is worth to note that the operator D^n is left inverse to J^n because $D^n J^n f(t) = f(t)$. On the other hand,

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{k=n-1} f^k(0^+)\frac{t^k}{k!} \tag{I7}$$

Application of the definition (I3) leads to:

$$D^\alpha t^y = \frac{\Gamma(y+1)}{\Gamma(y-\alpha+1)} t^{y-\alpha} \tag{I8}$$

and thus an expression of derivative basically identical to what is produced by (I3) is obtained. It should be noticed that the Riemann-Liouville fractional derivative of a constant is not a constant! Indeed, it accrues from (II7) that

$$D^\alpha c = \frac{c\, t^{-\alpha}}{\Gamma(1-\alpha)} \tag{I9}$$

For example, $D^{\frac{1}{2}} 1 = \frac{1}{\sqrt{\pi}\sqrt{t}}$.

Also, according to the Liouville fractional derivative:

$$D^\alpha_{-\infty} e^{\lambda t} = \lambda^\alpha e^{\lambda t}, \ \ \lambda>0 \tag{I10a}$$

However, it becomes

$$D^\alpha_{-\infty} e^{\lambda t} = t^{-\alpha} E_{1,1-\alpha}(\lambda t) \tag{I10b}$$

when the terminal is set to 0 (Riemann-Liouville), involving an E_t function (see Appendix II). Also:

$$D^\alpha_{-\infty} \sin \omega t = \omega^\alpha \sin\left(\omega t + \frac{\alpha\pi}{2}\right), \quad \text{for} \ \ \omega>0 \tag{I11a}$$

Whilst in ordinary calculus the derivative of an elementary function is also an elementary function (or combination), fractional derivatives need not follow this rule and they are often complicated, involving higher transcendental functions. For example, the fractional derivative of trigonometric functions becomes very long when the terminal is set to 0:

$$D^{\alpha}\sin\omega t = -\frac{1}{\Gamma(5-\alpha)}\left\{t^{1-\alpha}\,\omega\left[(-4+\alpha)(-3+\alpha)(-2+\alpha)F_{p,q}\left(\left(\frac{3-\alpha}{2},\,2-\frac{\alpha}{2}\right);\,-\frac{1}{4}t^2\omega^2\right)+\right.\right.$$
$$\left.\left.+2t^2\,\omega^2\,F_{p,q}\left(\left(\frac{5-\alpha}{2},\,3-\frac{\alpha}{2}\right);\,-\frac{1}{4}t^2\,\omega^2\right)\right]\right\} \quad \text{(I11b)}$$

Caputo fractional derivative

The Caputo definition of the fractional derivative is the most suitable for use in engineering problems because it requires initial conditions that are physically meaningful. It is defined by reversing the order between integration and differentiation in the Riemann-Liouville fractional derivative; i.e. one performs firstly a differentiation to the nearest integer order and the fractional integration follows. The resulting formula is:

$$D_*^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau$$
$$m-1<\alpha\le m \quad \text{(I11)}$$

We note that while for the Riemann-Liouville holds that $D^{n+\alpha} f(t) = \frac{d^n}{dt^n}\left[D^{\alpha} f(t)\right]$, for the Caputo the composition rule works in the reverse way $D_*^{\alpha+n} f(t) = D^{\alpha}\left[\frac{d^n}{dt^n} f(t)\right]$ i.e. <u>one needs to perform the integer differentiation first</u>.

The Laplace transform of the Caputo derivative is:

$$L\left[D_*^{\alpha} f(t)\right] = s^{\alpha}\,\hat{f}(s) - \sum_{k=0}^{m-1} f^{(k)}\left(0^+\right)s^{\alpha-1-k}$$
$$m-1<\alpha\le m \quad \text{(I12)}$$

The above summation indicates that for the calculation it suffices to know only the customary initial conditions $f(0^+)$, $f'(0^+)$ etc. Another strong point of the Caputo definition is that the fractional derivative of a constant is always zero.

Grünwald-Letnikov fractional derivative

A different approach to differentiation is to consider it as a limit of difference quotients, like the well-known $f'(t) = \lim_{h\to 0}\frac{f(t)-f(t-h)}{h}$. For the *m* order derivative this rule yields:

$$f^{(n)}(t) = \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k} f(t-k\,h) \quad \text{(I13)}$$

Considering values of *t* greater than a constant a_0 the above can be generalized for non integer order of differentiation to

$$D_{a_0^+}^{\alpha} f(t) = \lim\frac{1}{h^{\alpha}}\sum_{k=0}^{[(t-a)/h]}(-1)^k\binom{\alpha}{k} f(t-kh) \quad \text{(I14)}$$

Symbol [ ] means the integer part.

The above can be expressed as an integral, as follows (Podlubny 1999):

$$D_{a_0^+}^{\alpha} f(t) = \sum_{k=0}^{m-1}\frac{f^{(k)}(a_0)(t-a_0)\,t^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)}\int_{a_0}^{t}\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau \quad \text{(I15)}$$

It can be shown that one could arrive to the same result by starting from the Riemann definition of fractional derivative [expression (II4)]; i.e the two definitions are equivalent. In particular, for $t>0$ and $m-1<\alpha\le m$ the above becomes:

$$D_{0^+}^{\alpha} f(t) = \sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau \quad \text{(I16)}$$

The second term at the right-hand-side is the Caputo fractional derivative. Therefore the connection between the Grünwald-Letnikov, Riemann-Liouville and Caputo fractional derivatives becomes clear. The Grüwald-Letnikov definition is best suited for numerical calculation of the fractional derivative. More specifically in Podlubny (1999) is discussed that, for homogeneous initial conditions, the factors $w_k^{(a)} = (-1)^k\binom{a}{k}$, $k = 0,1,2...$ could be evaluated by the iteration:

$$w_0^{(a)} = 1;\quad w_k^{(a)} = \left(1-\frac{a+1}{k}\right)w_{k-1}^{(a)},\quad k = 1,2,3... \quad \text{(I17)}$$

APPENDIX II: SOLUTION THROUGH THE LAPLACE TRANSFORM

Omitting from (3) the excitation term, the Laplace transformed angle can be expressed as:

$$\Phi(s) = \frac{\varphi_0(Bs^{a-1}+Is)}{Is^2+Bs^{1+a}+mg\,\overline{GM}} \quad \text{(II1)}$$

The above holds under the assumption $\varphi(0) = \varphi_0$ and $\dot{\varphi}(0) = $. $\Phi(s)$ may be regarded as the product of two functions:

$$Z(s) = \frac{1}{Is^2+Bs^{1+a}+mg\,\overline{GM}} \quad \text{(II2)}$$

$$Q(s) = Is\varphi(0) + Bs^{a-1}\varphi(0) \quad \text{(II3)}$$

The reverse Laplace transforms of these functions are:

$$z(t) = L^{-1}\{Z(s)\} = \frac{1}{I}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{C}{I}\right)^k t^{2(k+1)-1}E^{(k)}_{1-a,2+(a+1)k}\left(\frac{-B}{I}t^{1-a}\right) \quad \text{(II4)}$$

In the above, $E_{a,\beta}(z)$ is the so called 2-parameter Mittag-Leffler function which should be discussed further. It simpler counterpart, the mono-parametric Mittag-Leffler function $E_a(z)$, is a generalization of the exponential function e^z and it is defined as:

$$E_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{II5}$$

On the other hand, the 2-parameter Mittag-Leffler is defined from the summation:

$$E_{a,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{II6}$$

The next properties can be easily proved:

$$E_{1,1}(z) = e^z \qquad E_{1,2}(z) = \frac{e^z - 1}{z} \qquad E_{1,3}(z) = \frac{e^z - 1 - z}{z^2}$$

and generally:

$$E_{1,m}(z) = \frac{1}{z^{m-1}} \left\{ e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right\} \tag{II7}$$

Returning to our equation:

$$q(t) = L^{-1}\{Q(s)\} = B t^{-(1+a)} \left(\frac{\varphi(0)}{\Gamma(-a)} \right) \tag{II8}$$

The roll angle should thus be produced by the convolution:

$$\varphi = z \circ q \tag{II9}$$

Given the complexity of functions $z(t), q(t)$, analytical calculation of the convolution integral seems to be straightforward only for some special values of the order *a*. Nonetheless, direct numerical solution of (3) is quite simple.