SIMILARITIES IN THE YAW AND ROLL DYNAMICS OF SHIPS IN EXTREME ASTERN SEAS

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1. Introduction

In large following waves there exist some interesting analogies between the dynamic behaviour of a steered ship in the yaw and roll directions. It is well known, for example, that capsize may occur due to fluctuations of the roll righting-arm [1]. Similar fluctuations in the stiffness term may take place also in yaw, originating from the combined effect of rudder control with the wave induced yaw moment. This may give rise to course instability which will be realized as deviation from the desired heading and broaching [2].

Consider a ship traveling in long following sinusoidal waves. In order to avoid coupling complications let us assume further that, due to high natural frequencies in heave and in pitch compared to the encounter frequency, the ship can maintain a state of quasi-static equilibrium on the vertical plane. If the waves are relatively steep, the geometry of the submerged part of the hull will vary noticeably, on the basis of the ship's position on the wave. This will be reflected in roll's righting-arm. Reduced or even negative roll restoring may arise when the middle of the ship is near to a wave crest, due to substantial "loss of waterline" (typically such a trend is more pronounced when there is low freeboard at midship combined with strong flare at the ends [3]). If roll restoring remains negative for sufficient time, so that heel finds the time to develop unopposed well beyond the "vanishing angle", then capsize due to the so-called pure-loss of stability mechanism will be realised [4]. In this case the magnitude of roll damping affects little the survivability of the ship. Capsize can occur of course also in a typical parametric resonance fashion and here damping will be much more important

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[5]. Practically, the variation of restoring must be however quite intensive, so that large amplitude roll can build-up within a small number of wave cycles.

The onset of yaw instability, in a similar wave environment, is a slightly more complex process because yaw is always coupled with sway. Furthermore, a control law for the rudder must be considered. Unlike with roll, if there is no active control in yaw, no restoring force exists in still water; but this may be created by the movement of the rudder which tends to bring the ship back on the correct course. If waves of length equal to the ship length or longer meet the ship from behind, they will create a moment that will be dependent on the angle between the direction of wave propagation and the ship's heading. This wave yaw moment works as a positive restoring component when the ship passes from crests (stabilizing effect). The opposite will be realized however in the vicinity of wave troughs where the waves will tend to orientate the ship perpendicular to the direction of their propagation. The relative magnitudes of the rudder's and the wave's moment will determine whether restoring becomes, in the region of the wave trough, negative. But even if it remains positive, a parametric mechanism with the potential to destabilize the horizontal-plane motion of the ship will have been set in place. The commonality of the underlying dynamics of yaw and roll is prevalent.

Our first objective in this paper is to identify the correspondence between yaw and roll parameters from the perspective of these Mathieu-type phenomena. Furthermore, we shall introduce an approach for assessing the effect of surge motion. As is well known, when the waves are large, the nonlinearity of surge cannot be neglected [6]. A manifestation of this nonlinearity is a virtual rescaling of time as the ship is spending longer time on the crests than on the troughs of the wave. In spite of the significance of this mechanism for the safety-critical motions, there has been no systematic analysis earlier on it.

2. Equations of motion for yaw and for roll

Consider the linear differential equations of sway and yaw [7], with the addition of wave excitation terms at their right-hand side:

$$\text{Sway:}\quad (m' - Y'_{\dot v})\dot v' - Y'_v v' + (m'x'_G - Y'_{\dot r})\dot r' + (m' - Y'_r) r' = Y'_\delta \delta + Y'_{(wave)} \qquad (1)$$

$$\text{Yaw:}\quad (m'x'_G - N'_{\dot v})\dot v' - N'_v v' + (I'_z - N'_{\dot r})\dot r' + (m'x'_G - N'_r) r' = N'_\delta \delta + N'_{(wave)} \qquad (2)$$

In the above v', r' are respectively sway velocity and yaw angular velocity, δ is the rudder angle, m' is ship mass and x_G is the longitudinal position of the centre of gravity; $Y'_{\dot{v}}$, $Y'_{\dot{r}}$, $N'_{\dot{v}}$, $N'_{\dot{r}}$ are acceleration coefficients (added masses/moments of inertia) and Y'_v, Y'_r, N'_v, N'_r, Y_δ are velocity coefficients (hydrodynamic damping terms). The wave's sway force and yaw moment are respectively, $Y'_{(wave)}$, $N'_{(wave)}$. The prime indicates nondimensionalised quantity and the overdot differentiation over time.

At first instance we shall assume that the yaw and sway velocities are restrained from building-up to high values (thus they remain small; as a matter of fact the resulting damping forces may be considered linear) through use of appropriate rudder control. Additionally, ship behaviour is examined at "some distance" from the region of surf-riding, so that, for this first part of the paper, it is not unrealistic to assume that surge velocity is constant.

We express the wave terms $Y'_{(wave)}$, $N'_{(wave)}$ in respect with the frequency of encounter (rather than as functions of absolute wave frequency and position). Also we neglect some phase difference (relatively to the wave) which might exist in these two types of excitation:

$$Y'_{(wave)} = Y'_w \, \psi \sin(\omega'_e t') \tag{3}$$

$$N'_{(wave)} = N'_w \, \psi \cos(\omega'_e t') \tag{4}$$

The following notation is applied: Y'_w, N'_w are wave force/moment coefficients; ψ is the ship's heading relatively to the wave ($\psi = 0$ when the sea is exactly following – generally, ψ is assumed small).

Consider further rudder control with a linear law based on two gains, k_1 and k_2: k_1 multiplies the instantaneous heading deviation from the desired course ψ_r, while k_2 multiplies yaw's angular velocity:

$$\delta = -k_1(\psi - \psi_r) - k'_2 r' \tag{5}$$

Substituting (3), (4) and (5) in (1) and (2), uncoupling yaw from sway and using well known expressions for system gain and time constants, K', T'_1, T'_2, T'_3 [7], a differential equation of heading angle with the following structure is obtained:

$$\psi'^{(3)} + b\ddot{\psi}' + p[1 + f\cos(\omega'_e t' - \sigma)]\dot{\psi}' + q^2[1 - h\cos(\omega'_e t' - \rho)]\psi = j \tag{6}$$

The above third-order differential equation has time-dependent coefficients in two places. As is well known however, if T'_1 is much greater than T'_2 and T'_3, we can use the so-called simplified yaw response model of Nomoto [8]. In that case the order of equation (6) is reduced by one:

$$T'\ddot{\psi}' + \dot{\psi}' = K'\delta + A'\psi\cos(\omega'_e t' - a) \tag{7}$$

K', T' are respectively system gain and time constants, ψ is relative heading angle (assumed small), δ is rudder angle, A' is wave excitation amplitude, ω'_e is the encounter frequency and a is a phase angle.

By coupling (7) with the autopilot equation (5) and dropping for simplicity the phase angle a, we obtain after some rearrangement:

$$\ddot{\psi}' + \gamma \dot{\psi}' + {\omega'_{0(yaw)}}^2 [1 - h\cos(\omega'_e t')]\psi = j \qquad (8)$$

In the above $\omega'_{0(yaw)} = \sqrt{k_1 K'/T'}$, $\gamma = (1 + k'_2 K')/T'$ (damping), $h = A'/k_1 K'$ (amplitude of parametric variation of restoring), $j = k_1 K' \psi_r / T'$. It is easily recognized that (8) is Mathieu's equation with the addition however of bias-like external static forcing term, j.

For stability, positive T' is required as $1/T'$ is the inverse of the damping of the unsteered vessel. However, large positive T' implies slow convergence towards the corresponding steady rate-of-turn which is determined by the value of the static gain K'. A trend exists for large T' to appear in conjunction with large K' which gives a nearly straight-line *spiral curve*. The effect of active control on damping is represented by the quantity $k'_2 K'/T'$. It depends thus on the yaw rate ("differential") gain term in the autopilot. If $T' < 0$, suitable choice of k'_2 can turn the damping of the system positive since k'_2 multiplies the positive quantity K'/T', thereby yielding stability for the steered ship in calm sea. The wave effects are lumped into the restoring and independent-periodic-forcing terms since the quantities K' and T' were assumed to be, at first approximation, unaffected by the wave. If the amplitude of wave excitation A' exceeds $k_1 K'$, then on the basis of (11) negative yaw restoring will arise around the trough. Should the duration of operation under negative restoring be long enough, undesired turning motion will be initiated ("broaching"). From a dynamics perspective there is complete equivalence with a capsize event of the so-called "pure-loss" type. It can be avoided if the proportional gain k_1 is chosen to be always greater than A'/K' even for the most extreme wave environment where the ship will operate (it should be a matter of further investigation to what extend this is technically feasible). A notable difference between the manifestation of this instability in roll and in yaw is that in roll it arises near the crest of the wave, whereas in yaw the ship becomes vulnerable near a trough.

We may rewrite (8) on the basis of heading error $\psi_1 = \psi - \psi_r$ and then apply the transformation $\tau = \omega'_{0(yaw)} t'$:

$$\frac{d^2\psi_1'}{d\tau^2}+2\zeta\frac{d\psi_1'}{d\tau}+[1-h\cos(\Omega\tau)]\psi=f\cos\Omega\tau \qquad (9)$$

where $\Omega=\omega_e'/\omega_{0(yaw)}'$ and $h=A'/(k_1\ K')$. Also, $f=A'\ \psi_r/(T'\ \omega_{0(yaw)}'^{\,2})$ which means that for $\psi_r=0$ the external forcing term of (9) will be zero. The damping ratio is given by the expression: $2\zeta=(1+k_2K')/\sqrt{k_1K'/T'}$ (the presence of k_1 inside ζ should be noted).

It is obvious from (9) that parametric instability of yaw may also arise, very much like that of roll. To establish the analogy we remind that the generic linearised in x equation of roll for a following sea is:

$$\frac{d^2\varphi'}{d\tau^2}+2\zeta\frac{d\varphi'}{d\tau}+[1-h\cos(\Omega\tau)]\varphi'=0 \qquad (10)$$

φ' is the normalised roll angle, $\varphi'=\varphi/\varphi_v$, with φ the true roll angle and φ_v the angle of vanishing stability. Although for the damping ratio, scaled time, amplitude of forcing and frequency ratio we have used the same symbols as in roll, the expressions from which we derive their values will be of course different in the yaw case. The damping ratio will be: $2\zeta=B\,\omega_{0(roll)}/(M\,g\,(GM))$ where B is the dimensional linear damping coefficient, M is ship mass and (GM) is the metacentric height. Also, $\Omega=\omega_e'/\omega_{0(roll)}'$ and $\tau=\omega_{0(roll)}\,t$, with roll's natural frequency given by $\omega_{0(roll)}=\sqrt{Mg\,(GM)/(I+\Delta I)}$. With substitution of $\omega_{0(roll)}$ in ζ we may obtain further: $2\zeta=B/\sqrt{(I+\Delta I)\,M\,g\,(GM)}$. Also, the amplitude of the parametric is $h=\delta(GM)/(GM)$ where $\delta(GM)$ is the difference in the values of metacentric height at the crest and in still water. This is a common assumption which may be sufficient for the preliminary character of this study but of course it results in a highly idealised formulation because the average (GM) has no reason to be identical with the still water (GM). In addition, the variation from trough to crest may not be sinusoidal.

2.1. CONDITIONS AT EXACT RESONANCE

For overtaking waves the frequency of encounter will be positive and for the case where no damping exists the condition of exact resonance will be: $\omega_e/\omega_0=2/\eta$, $\eta=1,2,3,\ldots$ (ω_0 may be the frequency of encounter of yaw or of roll). Thus with increasing η the vertices will tend to accumulate nearer to the zero frequency of encounter.

The expression of the encounter frequency for a following sea is $\omega_e = (2\pi/\lambda)(c-U)$. In yaw, time is commonly nondimensionalised on the basis of U/L (noted the resulting time-dependence). Therefore, the expression of the nondimensional frequency of encounter in yaw is: $\omega'_e = 2\pi L(c-U)/(\lambda U)$. With the substitutions $\omega'_e = 2\omega'_{0(yaw)}/\eta$ and $c/U = Fn_{wave}/Fn$ (Fn_{wave} is the Froude number corresponding to wave celerity) the parametric equation of the vertices of the corresponding undamped system is $Fn = Fn_{wave}/(1+\lambda\omega'_{0(yaw)}/\eta\pi L)$,. Given that $Fn_{wave} = \sqrt{\lambda/(2\pi L)}$ we may write further:

$$Fn = \sqrt{\lambda/(2\pi L)}/\left[1+\omega'_{0(yaw)}\lambda/(n\pi L)\right] \tag{11a}$$

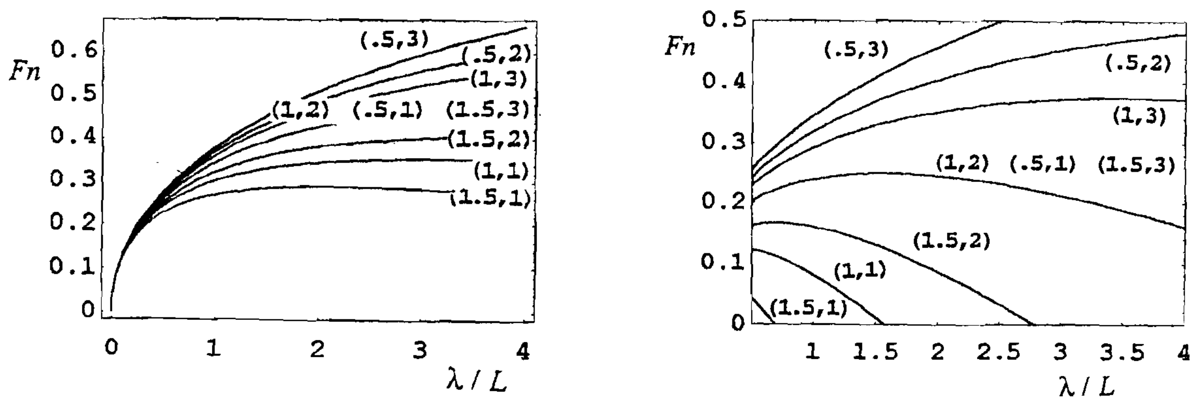
Consider further the domain of variation for the yaw natural frequency, which, as was found earlier, is expressed as: $\omega'_{0(yaw)} = \sqrt{k_1 K'/T'}$. It is known that for conventional ships, the ratio K'/T' usually takes values within the range $[0.3-1.4]$ (see for example [9]). It is derived that $\omega'_{0(yaw)}$ should lie in the range $[0.55\sqrt{k_1}$ - $1.18\sqrt{k_1}$]. With a proportional gain k_1 between 1.0 and 2.0, $\omega'_{0(yaw)}$ should then be between 0.55 and 1.67. In Fig. 1a is shown how the critical Fn would vary as function of the wave length-to-ship-length ratio λ/L, for three different values of $\omega'_{0(yaw)}$: 0.5, 1.0 and 1.5.

We shall consider now roll motion: The natural frequency is nondimensionalised on the basis of ship length and acceleration of gravity, $\omega'_0 = \omega_0\sqrt{L/g}$ (thus it is not speed dependent, at least in an explicit sense). The difference in the nondimensionalisation method between yaw and roll results in different parametric expressions of the critical Froude number:

$$Fn = \frac{\sqrt{\dfrac{\lambda}{L}}}{\sqrt{2\pi}} - \frac{\omega'_{0(roll)}\left(\dfrac{\lambda}{L}\right)}{\pi n} \tag{11b}$$

It is well known that container vessels are sometimes susceptible to parametric resonance, one of the reasons being that their operational speed falls near to the region of principal resonance which is the most dangerous. Extensive model tests have been carried out recently in Japan in order to identify the critical conditions for capsize. The occurrence of parametric instability was one of the investigated scenarios [10]. For an examined containership the measured natural frequency was $\omega'_{0(roll)} = 0.566$. In Fig. 1b are shown, for a range of roll natural frequencies, the critical Froude numbers for the first few resonances (as for the similar equation for yaw, damping is not included).

We should note that, unlike the parametric instability of roll which is well verified experimentally, for yaw little has been attempted so far on the experimental front.



(a) for yaw (b) for roll

Fig. 1: Critical Froude numbers for stability of idealised undamped system. We varied the pair (ω'_e, η) where ω'_e is the corresponding nondimensional natural frequency, and η is the order of the resonance.

2.2. MAGNITUDE AND EFFECT OF DAMPING

Even when a ship is equipped with bilge keels and fins, the damping ratio is usually quite low and very rarely goes above a value of, say, 0.2 (it should be noted however that ζ depends not only on the hydrodynamic characteristics of the hull but also on factors such as the metacentric height and the moment of inertia).For yaw on the other hand, the damping ratio depends strongly on the autopilot's gains. Common values are known to be in the range $0.8 < \zeta < 1.0$ [11]. Here lies therefore a very significant difference between the roll and yaw equations: The damping ratio of yaw is normally very large. This practically means that in order to be placed in a resonance region, the loss of yaw restoring at the trough should be very considerable. Usually the requirement implied by this is the existence of very steep waves. It is very interesting, and perhaps relevant, that about 40 years ago, during model experiments of broaching, it had been observed that as the encounter frequency departs from the zero value, the required wave steepness for broaching shows a very considerable increase [12].

While expressions of the stability boundary are not so difficult to find for relatively low damping, see for example [13], the same may not be said for the very large ζ appearing in the yaw equation. As a first indication of the effect of ζ on the critical h we may use the expression of Gunderson, Rigas & VasnVleck (1974) which is applicable for relatively large damping values:

$$h = \left(1-\zeta^2\right)\tanh\left(\pi 2\zeta\sqrt{\frac{\omega_0^2}{\omega_e^2}}\right) \tag{12}$$

For ζ as low as 0.3 the required h, according to (12), is 0.67 and 0.87 respectively for the principal and the fundamental resonance. One should bear in mind however that these values reflect long-term behaviour. For the build-up of significant motion within a small number of wave cycles (in a practical context this is most relevant) considerably higher values of h are required.

2.3. NONLINEARITY

In the roll equation nonlinearity exists in the restoring term (strong) and in damping (mild). Their effect is now relatively well understood, see for example [10, 13-16].

In yaw, nonlinearity is possible to appear in the damping term if the yaw velocity is allowed to become large (for example when the autopilot gain values are low). This relates with the S-shaped curve ("spiral curve") connecting the steady rate-of-turn with the angle of the rudder in still water for directionally unstable ships. It is common to take account of this nonlinearity through a cubic term of yaw velocity. After coupling with the autopilot equation we obtain the following nonlinear version of equation (8) which is left for future consideration:

$$\ddot{\psi}' + \gamma\,\dot{\psi}' - a\,\dot{\psi}'^3 + {\omega'_{0(yaw)}}^2[1 - h\cos(\omega'_e t')]\psi = j \tag{13}$$

3. Static and dynamic loss of stability

When roll stability in a following sea is examined, it is customary in naval architecture to distinguish between two mechanisms of capsize: (a) *Pure-loss of stability*, where the ship departs from the state of upright equilibrium due to negative restoring on a wave crest. Then, heel increases monotonically until the ship is overturned. In this mode the magnitude of damping plays little role. (b) *Parametric instability*, which is the classical Mathieu-type mechanism where the build-up is

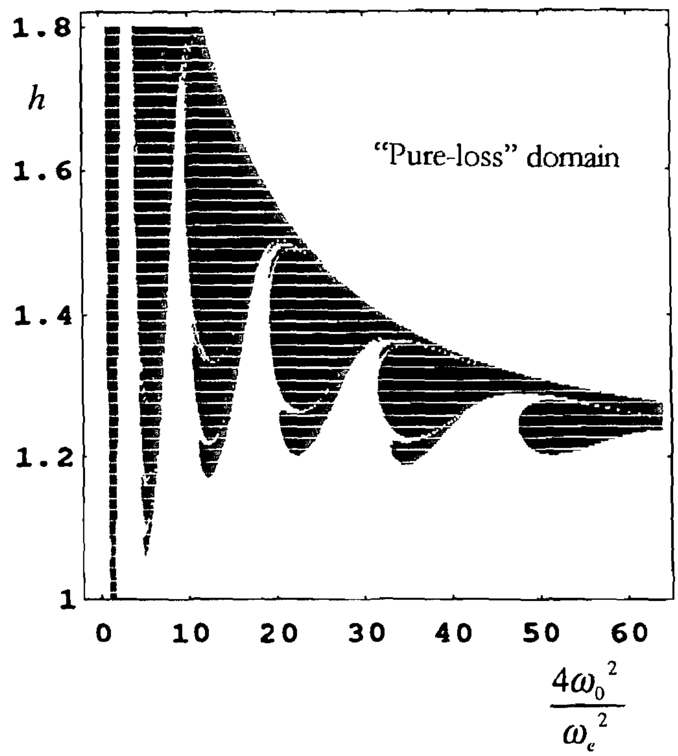


Fig. 2: A 'unifying' view of the domains of "pure loss" and of parametric instability for cubic-type restoring. "Pure loss" occupies the upper and left part of the parametric instability domain.

oscillatory and the magnitude of damping is very important. In Fig. 2 are shown the domains of pure loss and of parametric instability for a ship with a generic cubic-type restoring curve and with linear damping.

Instabilities of a similar nature are possible in yaw as well, resulting in broaching behaviour (sudden turn and deviation from the desired course). Especially the instability usually termed as *broaching due to surf-riding*, happening at Froude numbers near to the wave celerity, may be paralleled with the pure-loss mechanism[2].

A parametric-type mechanism of broaching also exists, which is more likely to happen at lower Froude numbers [12]. For this mechanism the discussion given in sub-section (2.2) is most relevant. Higher wave steepness is required for the occurrence of this instability due to the dominant effect of the large damping factor.

4. The effect of surge

We shall consider now the effect of surge motion for pure-loss and for parametric instability. An implicit assumption in our analysis so far, and also underpinning all earlier studies on ship parametric instability, has been that the forward speed may be assumed as constant. Such an assumption is not however always consistent with the wider context of the analysis. For dangerous dynamic behaviour of roll to arise, steep and long waves are required. Waves of this kind will incur also significant nonlinear effects on surge. The characteristic of large-amplitude surging is that it is asymmetric and the ship stays longer near the crests than near the troughs. This effect is imported into the yaw and roll dynamics through the restoring terms of the corresponding equations.

Consider the roll motion first: The nonlinearity of surge is detrimental for stability because around the crest (where the ship stays longer) restoring capability is reduced. For yaw on the other hand, the effect is opposite. Yaw stability is not worsened because the passage of the ship from the trough is quicker. The danger arises in steeper waves and especially during the process of capture in surf-riding, Fig. 3.

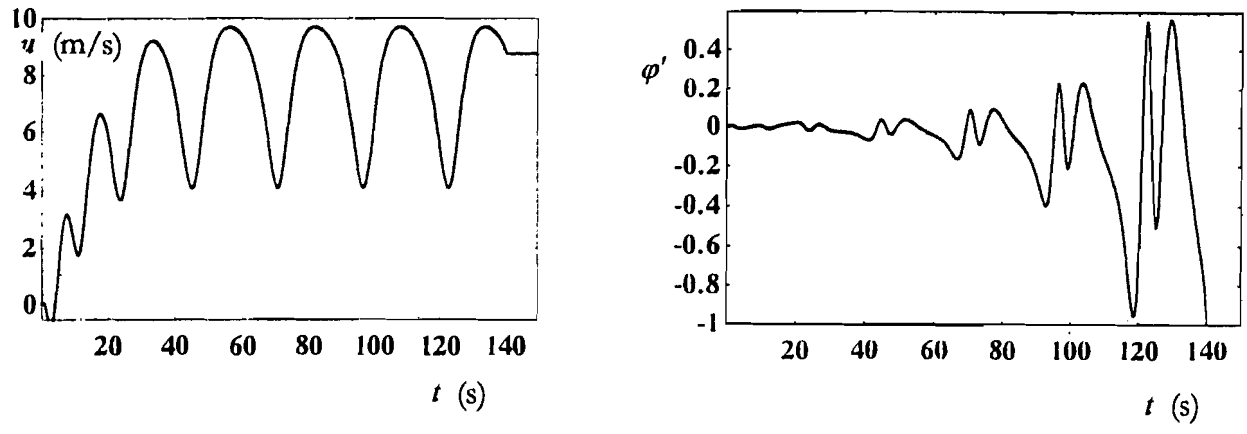


Fig. 3: Parametric instability and capsize due to large amplitude surging (very near to the boundary of surf-riding). Time-domain plots of surge velocity (left) and roll angle (right). The ship was initially with zero velocity at a crest.

The three main forces acting in the surge direction are the resistance, the wave and the propulsion force. As has been shown in [17] these forces result in the following differential equation for the surge motion:

$$\begin{aligned}(m - X_{\dot{u}})\frac{d^2x}{dt^2} + \{[3a_3c^2 + 2(a_2 - b_1)c + a_1] - b_2 n\}\frac{dx}{dt} + \\ +[3a_3c + (a_2 - b_1)](\frac{dx}{dt})^2 + a_3(\frac{dx}{dt})^3 + f\sin(kx) = \\ = b_1c^2 + b_2cn + b_3n^2 - (a_1c + a_2c^2 + a_3c^3)\end{aligned} \quad (14)$$

$-X_{\dot{u}}$ is the surge added mass, c is the wave celerity, n is the propeller's rate of rotation, x is the position of the ship on the wave measured from a moving system fixed on a wave trough; a_1, a_2, a_3 are the coefficients of the resistance polynomial. Likewise, b_1, b_2, b_3 are the thrust-related coefficients. The velocity u of the ship for an observer fixed on the earth is given from the relation $u = c - dx/dt$.

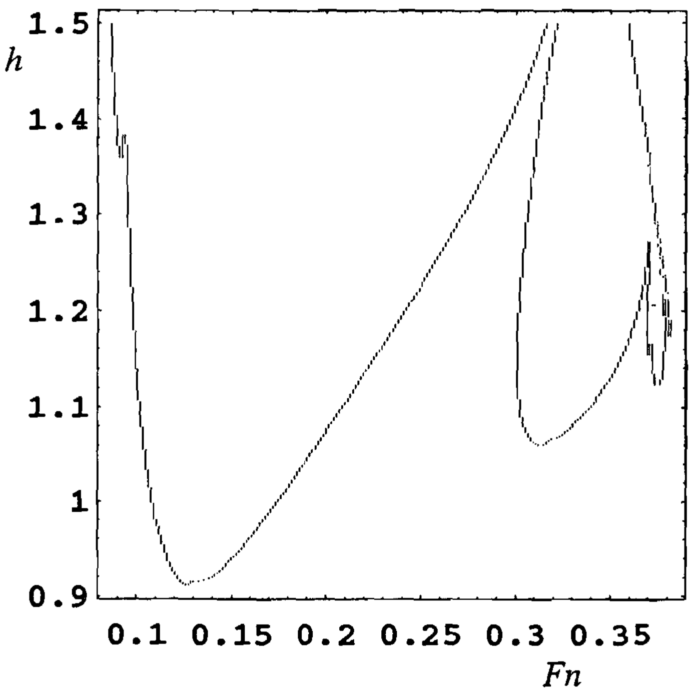


Fig. 4: Boundary lines of capsize due to parametric type instability when the nonlinear surge is taken into account. The upper-right boundary separates capsize from surf-riding. The other boundaries are interfaces with domains of ordinary periodic motion in surge.

Equation (14) is to be solved simultaneously with the equation of yaw or of roll, depending on whether the capsize or the broaching problem is considered. We have identified how the transition curves are modified when roll is coupled with surge. This coupling arises due to the existence of x in the restoring term of the roll equation:

$$\ddot{\varphi}' + 2\mu\,\dot{\varphi}' + \omega_0^2\left[1 - h\,\cos(kx)\right]\varphi' = 0 \qquad (15)$$

On the basis of the above equation we have found for what combinations of Fn and h the normalised roll angle φ' exceeded the value of 1.0 (from an initial perturbation 0.01 and with zero initial velocity) within a specified time ($t = 200$ s). The calculations were based on a ship with $\omega_{0(roll)} = 0.84$ ($\omega'_{0(roll)} = 1.577$) and $\mu = 0.0585$. Of course, having exceeded the value 1.0 does not necessarily mean capsize, since we may still lie inside the safe basin. But for a practical analysis this is a good basis for comparisons. Fig. 4 provides clear evidence that surge motion has a profound effect on the "capsize" domains. Further investigations are currently underway on this matter. For the considered ship, the principal resonance could not be realised in following waves because a negative Froude number is required for this (the ship should be backing rather than going forward). The lower part of the fundamental is the only place where there is some commonality with the conventional ('damped') Strutt diagram. The upper part of the fundamental has become considerably wider. The next resonance occupies an enlarged domain; but the two after this seem to degenerate. This may relate with the emergence of the surf-riding domain where the behaviour of the ship is stationary.

There, the ship will travel with the speed of the wave, having its middle located near to a trough.

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