DEVELOPING AN INTERFACE BETWEEN THE NONLINEAR DYNAMICS OF SHIP ROLLING IN BEAM SEAS AND SHIP DESIGN

K.J. Spyrou, B. Cotton & J.M.T. Thompson

Centre for Nonlinear Dynamics and its Applications
University College London
Gower Street, London WC1E 6BT, UK

ABSTRACT

The possibility to use in ship design certain recent results of the nonlinear analysis of beam-sea rolling in order to maximize resistance to capsize is discussed. The loci of transient and steady-state capsize are approximately located on the plane of forcing versus frequency through Melnikov analysis, harmonic balance and use of the variational equation. These loci can be parametrized with respect to the restoring and damping coefficients. The minimization of the capsize domain leads naturally to the formulation of an interesting hull optimization problem.

1. INTRODUCTION

Recent efforts to understand the mechanism of ship capsize in regular beam seas have revealed enormous complexity in large amplitude rolling response patterns, even though these investigations have relied on simple nonlinear, single-degree models [1]. Whilst the existence of bistability, jumps and subharmonic oscillations near resonance were known from earlier studies based on perturbation-like techniques (see for example [2], [3] on the forced oscillator; and [4] for a more ship-specific viewpoint) a whole range of new phenomena including global bifurcations of invariant manifolds, indeterminate jumps and chaos have been shown recently to underlie roll models with cubic or quartic potential wells. There are good reasons to believe that such phenomena are generic and their presence should be expected for a wide range of ship righting-arm and damping characteristics.

For the practising engineer this new information will be of particular value if it can be utilized effectively towards designing a safer ship. So far, rather than trying to discriminate between good and less good designs in terms of resistance to capsize in beam seas, the current analyses set their focus mainly on developing an understanding of the nature of the nonlinear responses in their various manifestations. However it seems that the time is now ripe for addressing also the design problem. Attempts to develop an interface between nonlinear analysis and ship design are by no means a novelty since they date back, at least, to the discussions about Lyapunov functions in the seventies and early eighties [5], [6]. Nonetheless, a meaningful and practical connection between nonlinear analysis and ship design is still wanting.

In our current research, the main ideas and some preliminary results of which are presented here, we are exploring the potential of two different assessment methods, based on well known approximate escape criteria of forced oscillators. The first method capitalizes upon the so-called Melnikov criterion which provides a fair estimate of the first heteroclinic tangency (homoclinic for an asymmetric system) that initiates erosion of the safe basin, Fig. 1 [7], [8], [9]. In the second method the key concept is the wedge-like boundary of steady-state escape on the *forcing* - versus - *frequency* plane [10], [11],

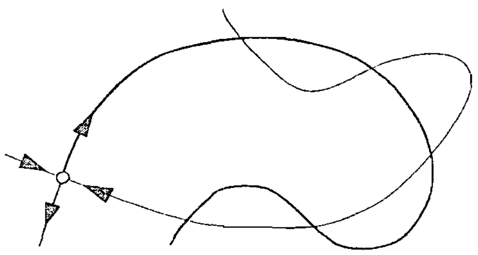


Fig. 1: Intersection of stable and unstable manifolds

[12]. The left branch of this boundary is the locus where jumps to capsize from the lower fold take place, Fig 2. As for the right branch, it is generally practical to assume as such the symmetry-breaking locus near resonance (or, the first flip for an asymmetric system).

These two criteria of transient and steady-state escape should be applied in conjunction with general-enough families of restoring and damping curves. A seventh-order polynomial is often seen as a suitable representation of restoring (see for example [13]). For damping, however, at this stage we shall confine ourselves to the equivalent linear one. Once the roll equation obtains a specific parametric form, expressions can be developed linking the coefficients of the restoring polynomial with damping, forcing and encounter frequency to the capsize loci. The obvious usefulness of these expressions is that they allow us to assess how hull modifications can affect the thresholds of transient or steady-state capsize. This leads to the setting up of an optimization process with governing objective the definition of a hull characterized by maximum resistance to capsize. The procedure offers also the interesting opportunity to evaluate the steady-state and transient criteria against each other, with the view to establishing whether they lead to similar optimum hull configurations.

2. KEY FEATURES OF THE SINGLE-WELL OSCILLATOR

Consider the following single-degree model for ship rolling, [1]:

$$\ddot{x} + D(\dot{x}) + R(x) = B + F\cos(\Omega\,\tau) \qquad (1)$$

where :

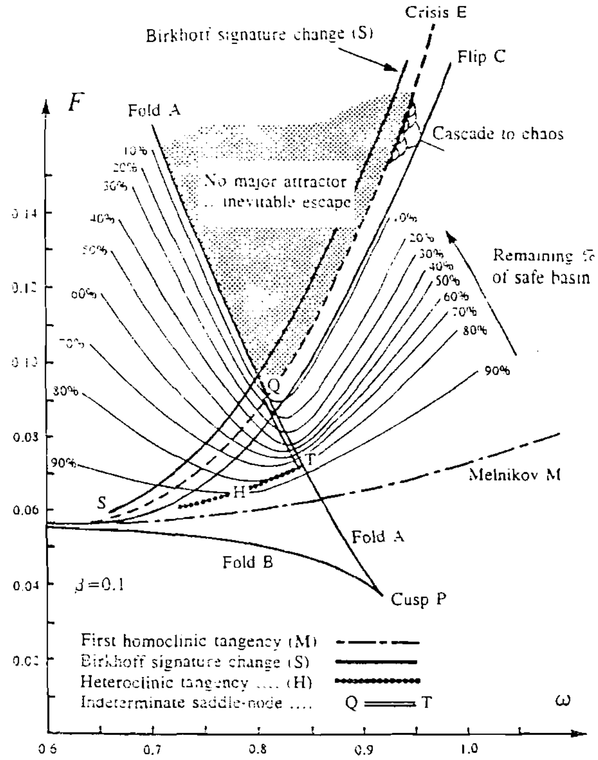


Fig. 2 : Bifurcation diagram of the escape equation, [14].

- x is the scaled roll angle, $x=\phi/\phi_v$,
- ϕ is the actual roll angle,
- ϕ_v is the angle of vanishing stability,
- $D(\dot{x})$ is the scaled damping function,
- $\Omega = \omega/\omega_0$,
- ω is the frequency of encounter between the ship and the wave (as we assume a beam-sea this is also the wave frequency),
- ω_0 is the natural frequency, $\omega_0=[W(GM)/I]^{1/2}$,
- W is the weight of the ship,
- (GM) is the metacentric height,
- I is the second moment of inertia including the added moment,
- F is the amplitude of the scaled external periodic forcing, $F= Ak\Omega^2/\phi_v$,
- Ak is the wave slope,
- B is a scaled constant excitation, for example due to steady wind,
- $R(x)$ is a scaled polynomial that approximates the restoring curve with $dR(x)/dx=1$ at $x=0$,
- τ is nondimensional time, $\tau=\omega_0 t$
- t is real time,

Let us consider for a while an asymmetric escape equation with periodic forcing, linear damping and a single quadratic, "softening" type, nonlinearity in restoring :

$$D(\dot{x}) = 2\,\zeta\,\dot{x} \quad , \qquad R(x) = x - x^2$$

This equation, which can be regarded as the simplest possible nonlinear equation akin to the capsize problem, has been studied to considerable depth, Figs. 2 and 3 [14]. Near resonance the response curve exhibits the well known bending-to-the-the left property that creates the lower fold A and the upper fold B. Point A is a saddle-node and a jump towards either some kind of resonant response or towards capsize will take place if the corresponding frequency threshold is exceeded. On the resonant branch different types of instability can arise. If the wave slope Ak is slowly increased, period-doublings (flips) are noticed that usually lead to chaos (a "symmetric" system with cubic instead of quadratic nonlinearity must first go through "symmetry-breaking" at a supercritical pitchfork bifurcation). Further increase in forcing leads ultimately to the so-called *final crisis*, where the chaotic attractor vanishes as it collides with a saddle forming a *heteroclinic chain.* At relatively high levels of excitation there is no alternative "safe" steady-state and subsequently escape is the only option. Long before such high levels of forcing have been attained, however, the "safe" basin has started diminishing after an homoclinic tangency (heteroclinic in the case of a symmetric system). The heteroclinic (homoclinic) tangency is usually considered as the threshold of transient escape. Melnikov analysis allows approximate analytical prediction of the relation between the oscillator's parameters on this threshold.

In a diagram of Ak versus Ω (for constant damping), the earlier discussed thresholds appear as boundary curves, Fig. 2. The locus of the first homoclinic tangency can lie at a considerable distance from the "wedge"-like boundary formed by the fold and symmetry breaking/period doubling loci. It is of course desirable that the Melnikov curve lies as high in terms of Ak as possible. It follows that a desirable hull configuration should present the minimum of its Melnikov curve at Ak as high as it can be. Alternatively, it is possible to take into account a range

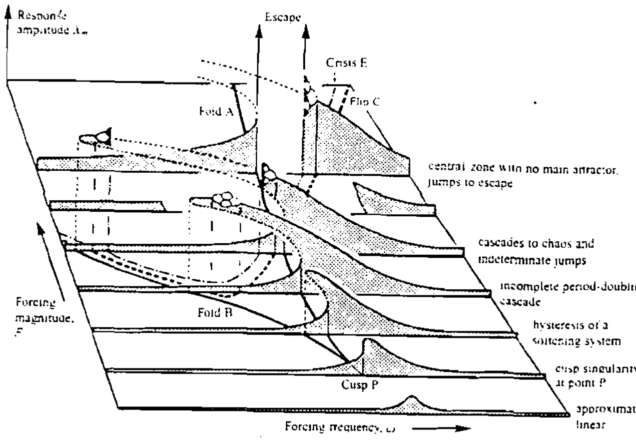


Fig. 3: Resonance response surface, [14]

rather than a single frequency, thus seeking to maximize the area below the Melnikov curve between some suitable low and high frequencies, respectively Ω_1 and Ω_2. In the ideal case where the Melnikov curve can be expressed explicitly as $Ak(\Omega)$, one will be seeking to identify the combination of restoring and damping coefficients, representing the connection with the hull, that maximizes the quantity

$$\int_{\Omega_1}^{\Omega_2} Ak(\Omega)\;\mathrm{d}\Omega$$

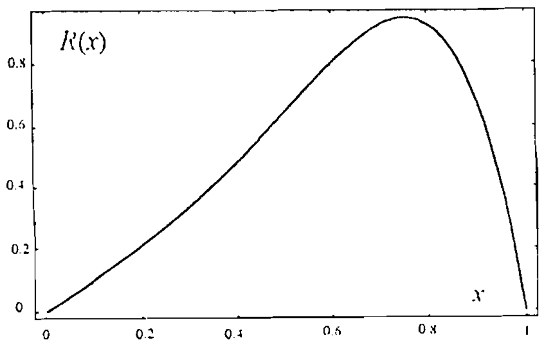
More sophisticated criteria based on wave energy spectra and thus incorporating probabilistic considerations could also be considered. These are left however for later studies.

A similar type of thinking can be applied for steady-state capsize. Here one could require the lowest point of the wedge to be as high as possible in terms of forcing; or again, the area under the wedge between suitable Ω_1 and Ω_2 to be maximized. One possible way of defining Ω_1 and Ω_2 rationally could be attained by drawing the breaking-wave line on the $(Ak,\ \Omega)$ plane and taking its intersections with the fold and flip curves. Unfortunately for the considered range of frequencies this line may not intersect the flip curve. The rational definition of Ω_1 and Ω_2 needs further consideration.

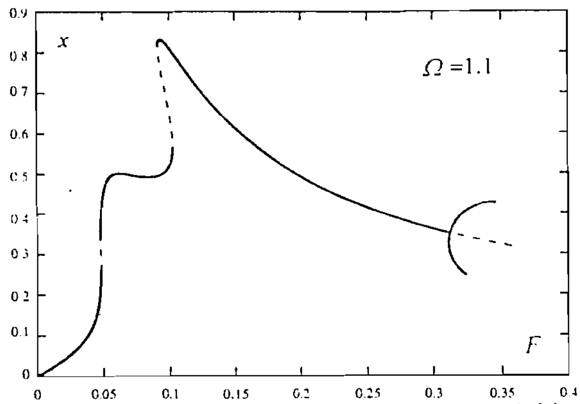
Assume finally the following "symmetric" representation of restoring :

$$R(x) = x + a_1 x^3 - a_2 x^5 + (-1 - a_1 + a_2)\, x^7 \qquad (2)$$

44



(4a)



(4b)

Fig. 4 : Restoring curve, (4a), and steady response curve, (4b), for a_1=1.5 and a_2=1

The main advantage in using the seventh-order polynomial is that it provides two points of inflection, see Appendix. Here a_1, a_2 are the two free parameters of the restoring curve. The coefficient of the seventh-order term is selected so that the saddle points are always at x=1 and -1. Thus we shall be dealing from now on with the following roll equation, Fig. 4:

$$\ddot{x} + 2\zeta\,\dot{x} + x + a_1 x^3 - a_2 x^5 + (-1-a_1+a_2)x^7 = F\cos(\Omega\,\tau) \qquad (3)$$

3. MELNIKOV-BASED CRITERIA

Details about Melnikov analysis can be found in a number of texts and no attempt will be made to repeat these here, e.g. [15], [16], [17]. The method is based on the calculation of the signed distance between the stable and unstable manifolds of one or more saddle equilibrium points when this distance is small. Melnikov analysis can also be regarded as an energy balance method where the total energy dissipated through damping should equal the energy supplied through the external forcing [14]. A more sophisticated version of the method can be applied also for highly dissipative systems [18].

Melnikov analysis includes basically the following stages. Firstly we calculate the Hamiltonian H of the unperturbed (ζ=F=0) system and from this the heteroclinic (homoclinic) orbit as $dx/d\tau = p(x)$. Then, we attempt to derive, if possible analytically, the time variation along this orbit: namely to derive expressions for x and $dx/d\tau$ that are functions of time, $x = h_1(\tau)$ and $dx/d\tau = h_2(\tau)$. This often represents the first major difficulty in applying the method. The next step is to calculate the Melnikov function:

$$M(\tau_0) = \int_{-\infty}^{+\infty} \mathbf{f}[\mathbf{x}(\tau)] \wedge \mathbf{g}[\mathbf{x}(\tau), \tau + \tau_0]\,d\tau \quad (4)$$

where, $\mathbf{x} = [x, dx/d\tau]^T$; $d\mathbf{x}/d\tau = \mathbf{f}[\mathbf{x}(\tau)]$ is the equation of the unperturbed system and the function $\mathbf{g}[\mathbf{x}, \tau]$ is periodic and represents the damping and forcing terms considered as constituting a perturbation. Also, τ_0 is phase with $0 < \tau_0 < 2\pi/\Omega$. The symbol means to take the cross product of vectors. The main objective in this method is to identify those marginal combinations of parameters where the Melnikov function admits real zeros.

Application for equation (3) :

<u>Unperturbed system</u> :

$$\ddot{x} + x + a_1 x^3 - a_2 x^5 + (-1-a_1+a_2)x^7 = 0 \quad (5)$$

which can be written in the form :

$$x_1 = x$$

$$\dot{x}_1 = x_2 = \partial H/\partial x_2$$

$$\dot{x}_2 = -x_1 - a_1 x_1^3 + a_2 x_1^5 - (-1-a_1+a_2)x_1^7 = -\partial H/\partial x_1 \quad (6)$$

Hamiltonian:

$$H = 0.5\,\{x_2^2 + x_1^2 + (a_1/2)\,x_1^4 - (a_2/3)\,x_1^6 + [(-1-a_1+a_2)/4]\,x_1^8\} \quad (7)$$

Heteroclinic orbit :

$$\dot{x} = \pm\ [(\frac{3}{4} + \frac{a_1}{4} - \frac{a_2}{12}) - x^2 - \frac{a_1}{2}x^4 + + \frac{a_2}{3}x^6 - \frac{-1 - a_1 + a_2}{4}x^8]^{1/2} \qquad (8)$$

Let the time variation along the heteroclinic orbit be : $x = h_1(\tau)$ and $dx/d\tau = h_2(\tau)$. These can be found with appropriate variable transformations, or they can be approximated.

Melnikov function :

$$M(\tau_0) = \int_{-\infty}^{+\infty} x_2\ \{F\cos[\Omega(\tau-\tau_0)] - 2\zeta\ x_2\} d\tau =$$
$$= F\ \cos(\Omega\ \tau_0) \int_{-\infty}^{+\infty} h_2(\tau)\cos(\Omega\tau)\ d\tau$$
$$- F\sin(\Omega\ \tau_0) \int_{-\infty}^{+\infty} h_2(\tau)\sin(\Omega\tau)\ d\tau -$$
$$- 2\zeta \int_{-\infty}^{+\infty} h_2^2(\tau)\ d\tau \qquad (9)$$

The second integral is expected to be zero because $h_2(\tau)\sin(\Omega\tau)$ is an odd function [$h_2(\tau)$ is expected to be even, $\sin(\Omega\tau)$ is of course odd]. However if the homoclinic orbit is considered it is the first integral that can be zero.

The condition to have simple zeros for the Melnikov function written in terms of Ak is thus:

$$Ak\Omega^2/\phi_v\ > 2\zeta\ \frac{\int_{-\infty}^{+\infty} h_2^2(\tau)\ d\tau}{\int_{-\infty}^{+\infty} h_2(\tau)\cos(\Omega\ \tau)\ d\tau} \qquad (10)$$

The threshold Ak that gives rise to equality in (10), Ak_{min} will mean tangent manifolds and will thus define the Melnikov curve $Ak=g(\Omega)$.

Criterion 1:

$Ak_{min}(\Omega)$ to become maximum in terms of the parameters a_1, a_2, ϕ_v, ζ. It is understood of course that as $2\zeta = b/[W(GM)I]^{1/2}$, where b is the true dimensional damping, (GM) and I participate also in the optimization.

Criterion 2 :

The following objective function S should be maximized:

$$S = \int_{\Omega_1}^{\Omega_2} Ak\ d\Omega =$$
$$= 2\int_{\Omega_1}^{\Omega_2} \zeta\ \frac{\int_{-\infty}^{+\infty} h_2^2(\tau)\ d\tau}{\int_{-\infty}^{+\infty} h_2(\tau)\cos(\Omega\tau)d\tau}\ d\Omega \qquad (11)$$

To make sure that the method produces meaningful alternative design solutions, additional conditions must be supplied. Current IMO or Naval GZ-curve shape criteria use as benchmarks the highest point of the curve as well as certain areas under the curve (up to 30 and 40 deg as well as between the two) see for example [19]. The search for maximum of the objective function should thus be constrained by suitable extra conditions that will guarantee that stability criteria in common use are being satisfied (see Appendix).

4. STEADY-STATE CRITERIA

These criteria require to locate the fold and symmetry breaking boundaries. Firstly, a low-order analytical solution of (3) is found with use of the method of harmonic balance. This solution is subsequently 'coupled' with suitable stability conditions. To identify the fold it is rather straightforward to request $\partial\Omega/\partial x_0 = 0$, where x_0 is the amplitude of roll motion, making sure of course that the lower fold A is the one considered. To approximate the locus of symmetry breaking we derive the variational equation and we find the relation that allows the existence of an asymmetric solution (or of a subharmonic solution in the case of an asymmetric system).

(a) Solution with harmonic balance

We rewrite (3) as follows :

$$\ddot{x} + 2\zeta\ \dot{x} + x + a_1x^3 - a_2x^5 + (-1 - a_1 + a_2)x^7 =$$
$$= F\cos(\Omega\tau - \theta) \qquad (12)$$

where θ is the phase difference between excitation and response that must be identified. We seek a steady-state solution $x = x_0\cos(\Omega\tau)$. We substitute this into (12), expand the trigonometric terms, retain only the terms of harmonic frequency and equate the coefficients of $\cos(\Omega\tau)$ and

$\sin(\Omega\tau)$ on both sides of the equation, obtaining finally:

$$x_0 = \frac{F}{(N^2 + 4\zeta^2 \Omega^2)^{1/2}} \qquad (13)$$

$$\theta = \arctan\left(\frac{-2\zeta\,\Omega}{N}\right) \qquad (14)$$

where

$$N = -\Omega^2 + M \qquad (15)$$

$$M = 1 + \frac{3a_1 x_0^2}{4} - \frac{5a_2 x_0^4}{8} + \frac{9(-1 - a_1 + a_2)x_0^6}{16} \qquad (16)$$

An alternative useful form of the above is obtained by solving for Ω :

$$\Omega = \frac{\sqrt{2}}{2}\sqrt{(2M - 4\zeta^2) \mp} \\ \mp\sqrt{(2M - 4\zeta^2)^2 - 4(M^2 - F^2/x_0^2)} \qquad (17)$$

With plus we obtain the high frequency branch and with the minus the low one.

(b) Approximation of the fold

With differentiation of (13) in terms of x_0, imposition of the condition $\partial\Omega/\partial x_0 = 0$ and some rearrangement, the following relation is derived:

$$\Omega^4 - (2M + x_0 M' - 4\zeta^2)\Omega^2 + (M^2 + M M' x_0) = 0 \qquad (18)$$

An alternative expression based on F can also be derived :

$$F^4 - x_0^3 M'(x_0 - 4\zeta^2) F^2 + 4M\zeta^2 = 0 \qquad (19)$$

where

$$M' = \partial M / \partial x_0 = \frac{a_1 x_0}{2} - \frac{5a_2 x_0^3}{2} + \\ + \frac{27(-1 - a_1 + a_2)x_0^5}{8} \qquad (20)$$

Finally x_0 must be eliminated between (17) and (18) and also F must be written in terms of Ak to obtain an expression, say $G(Ak,\Omega) = 0$, that defines the fold locus on the (Ak,Ω) plane.

(c) Approximation of the symmetry breaking locus

Consider again (3) and let x be increased by a very small amplitude ξ, such that ξ^2, ξ^3 etc. can be neglected. Then by substituting x with $x + \xi$ in (3) we obtain:

$$[\ddot{x} + 2\zeta\dot{x} + x + a_1 x^3 - a_2 x^5 + (-1 - a_1 + a_2)x^7 - \\ F\cos(\Omega\tau)] + \ddot{\xi} + 2\zeta\dot{\xi} + [\partial q(x)/\partial x]\,\xi = 0 \qquad (21)$$

where $q(x) = R(x) - F\cos(\Omega\tau)$ (22)

In (21) the quantity inside the first brackets is zero by definition and therefore we are left only with the so-called *variational equation* [20], [21] :

$$\ddot{\xi} + 2\zeta\dot{\xi} + [\partial q(x)/\partial x]\,\xi = 0 \qquad (23)$$

$$\ddot{\xi} + 2\zeta\dot{\xi} + [1 + 3a_1 x^2 - 5a_2 x^4 + \\ + 7(-1 - a_1 + a_2)x^6]\,\xi = 0 \qquad (24)$$

where $x = x_0\cos(\Omega\tau)$. We want to find the threshold where an asymmetric solution first appears, so we consider a perturbation ξ that includes constant term and second harmonic :

$$\xi = b_0 + b_{2c}\cos(2\Omega\tau) + b_{2s}\sin(2\Omega\tau) \qquad (25)$$

Parenthetically is mentioned that if the asymmetric equation was used we should consider a subharmonic perturbation :

$$\xi = b_{1c}\cos[(\Omega/2)\tau] + b_{1s}\sin[(\Omega/2)\tau] + \\ + b_{3c}\cos[(3\Omega/2)\tau] + b_{3s}\sin[(3\Omega/2)\tau] \qquad (26)$$

With substitution of x and ξ [from (25)] in (21) and application of harmonic balance, where we retain only terms up to second harmonic, we obtain a linear system of algebraic equations in terms of b_0, b_{2c} and b_{2s} :

Coefficient of the constant term

$$[1+\frac{3}{2}a_1x_0^2-\frac{15}{8}a_2x_0^4-\frac{35}{16}(1+a_1-a_2)x_0^6]\,b_0+$$
$$+[\frac{3}{4}a_1x_0^2-\frac{5}{4}a_2x_0^4-\frac{105}{64}(1+a_1-a_2)x_0^6]\,b_{2c}+$$
$$+\quad 0\,b_{2s} \qquad = 0 \qquad (27)$$

Coefficient of $\cos(2\Omega\tau)$

$$[\frac{3}{2}a_1x_0^2+\frac{5}{2}a_2x_0^4-\frac{105}{32}(1+a_1-a_2)x_0^6]\,b_0+$$
$$+[1+\frac{3}{2}a_1x_0^2-\frac{35}{16}a_2x_0^4-$$
$$-\frac{91}{32}(1+a_1-a_2)x_0^6-4\Omega^2]\,b_{2c}+$$
$$+(4\zeta\Omega)\,b_{2s} \qquad = \qquad 0 \qquad (28)$$

Coefficient of $\sin(2\Omega\tau)$

$$(4\zeta\Omega)\,b_0-$$
$$-(4\zeta\Omega)\,b_{2c}+$$
$$+[1-\frac{3}{2}a_1x_0^2-\frac{25}{16}a_2x_0^4-\frac{49}{32}(1+a_1-a_2)x_0^6-$$
$$-4\Omega^2]\,b_{2s}=0 \qquad (29)$$

The condition $\Delta=0$ where Δ is the determinant of (27), (28) and (29) provides the sought equation for the symmetry-breaking locus. It is interesting that the expression is analytically solvable for Ω. Again however the elimination of x_0, through combining with (17), is problematic.

(d) Derivation of steady-state criteria

The lowest point of the wedge corresponds obviously to the intersection of the curves $G(Ak,\Omega)$ and $\Delta(Ak,\Omega)=0$. Let us define this point as (Ak_0,Ω_0). We want to maximize Ak_0 in terms of the coefficients a_1, a_2, ϕ_v and also ζ [which, it should not be forgotten, includes *(GM)*]. Also in respect to the area criterion, if $Ak_G(\Omega)$, $Ak_\Delta(\Omega)$ are explicit representations of wave slope in terms of Ω at the fold and flip loci respectively, we want:

$$\int_{\Omega_1}^{\Omega_0} Ak_G(\Omega)\mathrm{d}\Omega \;+\; \int_{\Omega_0}^{\Omega_2} Ak_\Delta(\Omega)\,\mathrm{d}\Omega$$

to be maximum.

5. STEADY VERSUS TRANSIENT CAPSIZE CRITERIA

Although the transient and steady-state capsize criteria are dynamically different and the basin erosion begins much earlier than the first period doubling, it is not known how they reflect on the actual optimization parameters. Do they result in similar optima or do they produce considerably different ones ? With the earlier developed tools it should be possible to infer to what extent the steady-state and capsize criteria coincide in their predictions of the optimum hull configuration. It is hoped that it will be possible to provide specific answers in a future publication.

6. REFERENCES

1. Thompson, J.M.T. Designing against capsize in beam seas: Recent advances and new insights, *Applied Mechanics Reviews*, **50**, 5, 1997, pp. 307-325.

2. Stoker, J.J. : *Nonlinear Vibrations in Mechanical and Electrical Systems*, 1950, Wiley, New York.

3. Nayfeh A.H & Mook, D.T. : *Nonlinear Oscillations*, 1979, Wiley, New York.

4. Cardo, A. Francescutto, A. & Nabergoj, R. : Ultraharmonics and subharmonics in the rolling motion of a ship: Steady-state solution. *International Shipbuilding Progress*, **28**, 326, 1981, pp. 234-251.

5. Odabashi, A.Y.: Conceptual understanding of the stability theory of ships, Schiffstechnik, **25**, 1978, pp. 1-18.

6. Caldeira-Saraiva, F.: A stability criterion for ships using Lyapounov's method, *Proceedings*, The Safeship Project, June 1986, Royal Institution of Naval Architects, London.

7. Thompson, J.M.T., Rainey R.C.T & Soliman, M.S.: Ship stability criteria based on chaotic transients from incursive fractals. *Philosophical Transactions of the Royal Society of London* A(1990) **332**, pp. 149-167.

8. Kan M.: Chaotic capsizing, *Proceedings*, ITTC SKC-KFR Meeting on Seakeeping Performance, September 1992, Osaka, pp. 155-180.

9. Falzarano, J.M., Shaw, S.W., Troesch,A: Application of global methods for analysing dynamical systems to ship rolling motion and capsizing, *International Journal of Bifurcation and Chaos*, **2**, 1, 1992, pp. 101-115.

10. Szemplinska-Stupnicka, W.: The refined approximate criterion for chaos in a two-state mechanical oscillator, *Ingenieur-Archiv*, **58**, 1988, Springer-Verlag, pp.354-366.

11. Szemplinska-Stupnicka, W.: Cross-well chaos and escape phenomena in driven oscillators. *Nonlinear Dynamics*, **3**, 1992, pp. 225-243.

12. Virgin, L.N.: Approximative criteria for capsize based on deterministic dynamics, *Dynamics and Stability of Systems*, **4**, 1, 1989, pp. 55-70.

13. Strathclyde Stability Program, University of Strathclyde, Department of Ship & Marine Technology, 1984.

14. Thompson, J.M.T.: Global dynamics of driven oscillators: Fractal basins and indeterminate bifurcations. Chapter 1 of *Nonlinear Mathematics and its Applications*, 1996, P.J. Aston(ed.), Cambridge University Press, pp. 1-47.

15. Guckenheimer, J. & Holmes, P.J.: *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, 1983, Springer-Verlag, Applied Mathematical Sciences, **42**, New York.

16. Birkdash, M.U., Balachandran, B., Nayfeh, A.H.: Melnikov analysis for a ship with a general damping model, *Nonlinear Dynamics*, **6**, 1994, pp. 101-124.

17. Nayfeh, A.H. and Balachandran, B.: *Applied Nonlinear Dynamics: Analytical, Computational and Experimental Methods*, 1995, Wiley Series in Nonlinear Science, New York.

18. Salam, F.: The Melnikov technique for highly dissipative systems. *SIAM Journal of Applied Mathematics*, **47**, 1987, pp. 232-243.

19. Stability standards for surface ships, Naval Engineering Standard 109, Ministry of Defence, Sea Systems Controllerate, Issue 3, August 1989, Bath, UK.

20. Hayashi, C.: *Nonlinear Oscillations in Physical Systems*, 1964, McGraw Hill, New York.

21. McLachlan, N.W.: *Ordinary Nonlinear Differential Equations in Engineering and Physical Sciences*, 1956, Oxford at the Clarendon Press.

APPENDIX

Consider the following polynomial for restoring:

$$R(x) = x + a_1 x^3 - a_2 x^5 + (-1 - a_1 + a_2)\, x^7$$

Area under the curve :

$$\int_0^1 R(x)\, dx = \frac{9 + 3a_1 - a_2}{24}$$

The 'true' area under the $GZ(\phi)$ curve is :

$$W\ (GM)\phi_v^{\ 2}\ \frac{9 + 3a_1 - a_2}{24}$$

The area up to an angle ϕ is :

$$W\ (GM)\ [\ \frac{\phi^2}{2} + \frac{a_1\,\phi^4}{4\phi_v^2} - \frac{a_2\,\phi^6}{6\phi_v^4} + \frac{(-1 - a_1 + a_2)\ \phi^8}{8\phi_v^6}\ ]$$

The maximum of the curve is found by solving for x the equation $dR(x)/dx = 0$:

$$(x^2)^3 - \frac{5\,a_2}{7(-1-a_1+a_2)}(x^2)^2 + \frac{3\,a_1}{7(-1-a_1+a_2)}(x^2) + \frac{1}{7(-1-a_1+a_2)} = 0$$

There is one real and positive root which can be found analytically with, for example, *Mathematica* . For the equation

$$(x^2)^3 - a\,(x^2)^2 + b\,(x^2) + c\,x = 0$$

the real and positive root is :

$$x_{max} = \sqrt{\left[\frac{a}{3} - \frac{2^{1/3}(-a^2+3b)}{3D} + \frac{D}{3\ 2^{1/3}}\right]}$$

where :

$$D = \{2\,a^3 - 9\,a\,b + \sqrt{[4(-a^2+3b)^3 + (2a^3 - 9ab - 27c)^2]} - 27c\}^{1/3}$$

Points of inflection at $d^2 R(x)\ /\ dx^2 = 0$:

$$x_{infl} = \sqrt{\frac{10\,a_2 \pm \sqrt{100\,a_2^2 - 252\,a_1(-1-a_1+a_2)}}{42\,(-1-a_1+a_2)}}$$