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Design Criteria for Parametric Rolling

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ABSTRACT

Parametric rolling is a phenomenon of instability of a ship's upright state that can be realised in a longitudinal seaway and often leads to an oscillatory roll motion with moderate or large amplitude. In recent years it has been brought to the centre of attention following an accident and a discussion has been opened about the possibility of the adoption of relevant design criteria. Among ship parameters, roll damping has been singled out as the key factor governing the propensity toward this behaviour. This paper presents a review of the state of the art covering the various facets of parametric rolling for a deterministic, as well as for a probabilistic, environment. Furthermore, some new ideas about the development of practical design criteria are presented, based on the interfacing of deterministic transient responses with the probabilistic characteristics of wave groups.

1. INTRODUCTION

The problem of parametric rolling behaviour of ships in waves has attracted considerable interest recently following the accident of a C11-class containership, which has been described as the most costly containership casualty in history [France *et al.* 2003]. Attention was focussed on the propensity of large post–Panamax containerships for parametric rolling in "head-seas", the condition where the accident was reported to have occurred. However, parametric rolling has been traditionally linked with ship operation in following-seas where for several ship types it is easier, given a relatively low metacentric height, to satisfy one of the key conditions of parametric rolling; that is, the period of the waves as encountered by the ship to be near to one half of the natural roll period. IMO's [1995] "guidance to the master" (which notably refers to following-seas only) contains a recommendation for detecting the onset of parametric rolling during operation: ship masters are advised to determine the wave period through observation, transform it to encounter period on the basis of a suitable diagram that takes into account speed and heading, and then compare it to the one half of the natural roll period (as well as to the natural period itself, for the avoidance of "synchronous" rolling). Susceptibility to parametric rolling also depends, however, on the degree of variation of roll restoring moment between wave crests and troughs. The required amplitude of variation is determined principally by roll damping, by the wave "groupness" of the seaway, and by the run length of the encountered wave groups. Roll damping may be singled out as the key design parameter, determining the extremity of the environment where parametric rolling could be realised.

Whilst well-known as a phenomenon for at least half a century [Grim 1952; Kerwin 1995; Arndt & Roden 1958; Pauling & Rosenbersg 1959], no specific design requirements referring to parametric rolling have yet found their way into the IMO stability regulations. A possible explanation is that, while it is often the cause of intensive rolling, it is rarely documented to lurk behind a specific capsize accident. Yet, the market is becoming gradually conscious of the fact that even 'non capsizal' instabilities could be responsible for tremendous effects in terms of loss or damage of property and business interruption. Furthermore, the large number of containers lost overboard every year, according to a source between 2,000 and 10,000, represents a serious hazard for smaller vessels [Roenbeck 2003]. It is indicative of current interest that articles about parametric rolling appear often in the daily maritime press (e.g. Gray [2001]; Tinslay [2003]). A classification society has recently taken the lead, publishing a technical guide for the parametric rolling of containerships [ABS 2004].

A review of recent literature leads to the conclusion that, by-and-large, the dynamics of parametric rolling are nowadays well understood [Blocki 1980; Spyrou 2000 & 2004; Neves 2002; Bulian *et al* 2003; Umeda *et al* 2003; Shin *et al.* 2004].

Design aids, ranging between simple analytical formulae and complex numerical simulation codes, can be used for ruling out, or at least for containing, the probability of displaying parametric rolling. Unfortunately, as the process of ship design continues to be primarily regulations-led, this wealth of knowledge seems to be little utilised by the designers. As a matter of fact, some ships like modern post Panamax containerships and probably some of the new large passenger ships, especially those characterised by a heavily flared bow and flat stern with wide transom, may be sailing without having examined their tendency to display parametric rolling in a longitudinal seaway. Dedicated experiments like those of Dalinga *et al.* [1998] centred on a typical cruise ship and those reported by France *et al.* [2003] for the C11 containership seem to substantiate the concern.

In the present paper the intention is to demonstrate what current theories could offer in terms of prediction of parametric rolling. Deeper issues, such as the effect of nonlinearities, new phenomena due to coupled motions (especially the effect of heave/pitch in head seas and the interference of surging in following seas) and, last but not least, parametric rolling in a probabilistic context are also reviewed. As a resonance phenomenon by nature, parametric rolling calls, in the first instance, for a "deterministic" treatment where the effect of the environment is assumed as basically periodic. But of course, none could disregard that the seaway is stochastic. A meaningful interfacing of the deterministic and probabilistic facets of the problem is very desirable as it is can help to set the right level of stringency for the design requirements that accrue from the application of the various theories. Yet the issue is still scientifically unsettled. A methodology that promises to bring these two together within a single assessment procedure is also outlined in the paper.

2. BASIC "DETERMINISTIC" ANALYSIS

Parametric rolling is a resonance phenomenon manifested by the sudden oscillatory growth of roll which develops despite the absence of wave excitation in the direction transverse to the ship. It is broadly perceived to occur if the ratio of natural roll frequency, ω_0, to the encounter frequency, ω_e, satisfies the following resonance condition

$$2\frac{\omega_0}{\omega_e} \cong \pm n, \quad n = 1,2,3,... \tag{1}$$

The above condition defines, in fact, the vertices of the instability regions of Mathieu's equation. Basic facts about the instability regions of Mathieu systems, as well as their mathematical description, can be found in standard mechanics textbooks like Nayfeh & Mook [1979] and Hayashi [1985]. Given that $\omega_e = k(c-U)$, with $k = 2\pi/\lambda$, $c = \sqrt{gk}$, where k is wave number, c is wave celerity and λ is wave length, it is possible to convert equation (1) to an expression of Froude number. For positive frequencies of encounter this yields the following useful expression

$$Fn = \left(\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda}{L}} - \frac{2}{nT_0'}\frac{\lambda}{L}\right) \tag{2}$$

where L is the ship length and $T_0' = T_0\sqrt{g/L}$ is the non-dimensional natural roll period. With all other factors assumed unaffected, if the ship were sailing in an oblique sea the Froude number should be divided by $\cos\psi$ (with $\psi = 0$ corresponding to a following sea). This happens when the waves overtake the ship and also in all 'head sea' scenarios. A plot of equation (2) for a containership is shown in figure 1. The case $n = 1$ is the most likely to be realized and, in general, it dominates the attention in the literature.

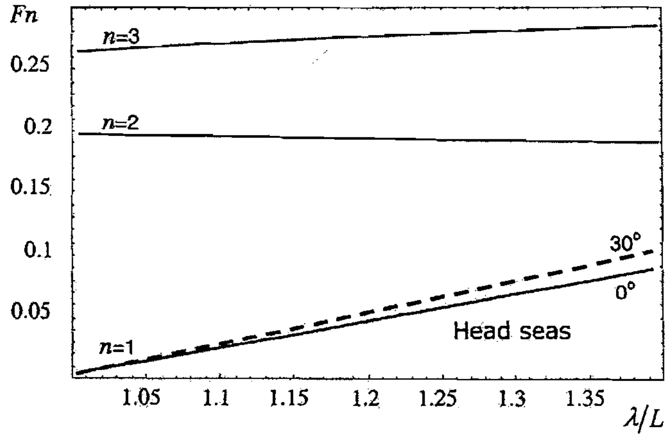

Figure 1. Froude numbers that could give rise to exact resonance for a containership. (L_{BP} = 262.0 m and T_0 = 25.7 s).

If the ship is prone to large variations of metacentric height (GM) from crest to trough, then encounter frequencies that depart from the exact resonance condition $\omega_e = \pm(2/n)\omega_0$ could still be eligible for producing growth of roll. As a matter of fact, it suffices to satisfy equation (1) or (2) only in an approximate sense. We shall scale the amplitude of GM variation against the average GM of the ship on the wave. In order to simplify the analysis the latter could be assumed, in the first instance, to be approximately equal to still water GM. Of course, strictly speaking there is no need for having the still water GM to coincide with the GM at the middle of the wave's up-slope or down-slope and, if a detailed calculation of the critical GM fluctuation is underway, the average GM on the considered wave should be used for the scaling (which means of course that the scale will vary with wave length and steepness). Taking things further, there also isn't any need for having a harmonic fluctuation of GM from crest to trough even if the wave were assumed to be perfectly harmonic. Strictly speaking this effect should be periodic and an analysis in Fourier series is likely to show several multiple frequencies in the fluctuation of GM. As a matter of fact, a so-called Hill's equation would be perhaps more representative. Whilst a direct attack on the problem in this spirit is nowadays possible, in the present context it would not help us to acquire the basic understanding about the phenomenon. Under the assumptions of 'small' and harmonically varying h ($= \frac{\delta GM}{GM} = \frac{GM_{trough} - GM_{crest}}{2GM}$), a well-known result from perturbation analysis for the boundary line of the first (principal) resonance is (see for example Hayashi [1985])

$$h_{1+} = 2\left(1-\frac{1}{a}\right) \quad h_{1-} = 2\left(\frac{1}{a}-1\right) \tag{3}$$

where the first equation refers to $a \geq 1$ and the second to $a<1$. The independent parameter a is defined as $a = 4\omega_0^2/\omega_e^2$. To grasp the order of magnitude, an ω_e higher than the frequency of exact principal resonance (= $2\omega_0$) by 10%, would entail, according to equation (3), h = 0.42. On the other hand, for an ω_e 10% lower than $2\omega_0$ the required h becomes 0.3471.

3. RATE OF GROWTH OF ROLL IN THE FIRST REGION OF INSTABILITY

For a "Mathieu system" the unstable motion that corresponds to the first region of instability should build up according to the following approximate general solution [Hayashi 1985]

$$\varphi(t) \approx c_1 e^{\mu\omega_0 t}\sin(\omega_0 t - \sigma) + c_2 e^{-\mu\omega_0 t}\sin(\omega_0 t + \sigma) \tag{4}$$

where μ, σ, are functions of a, h and they are determined from the relationships (only first-order terms are kept)

$$\cos 2\sigma \approx \frac{2(a-1)}{ah} \quad \left(-\frac{\pi}{2} \leq \sigma \leq 0\right), \qquad \mu \approx -\frac{\sqrt{a^2h^2 - 4(a-1)^2}}{4} \tag{5}$$

At a = 1 the coefficient μ obtains its maximum value μ_{max} = $-h/4$ where σ = $-\pi/4$. We can then determine the corresponding growth per roll cycle

$$\frac{\varphi(T_0)}{\varphi(0)} = \frac{\frac{\sqrt{2}}{2}\left(c_1 e^{\frac{\pi h}{2}} - c_2 e^{-\frac{\pi h}{2}}\right)}{\frac{\sqrt{2}}{2}(c_1 - c_2)} \tag{6}$$

After substitution of the initial conditions $\varphi(0) = \varphi_0$ and $\dot{\varphi}(0) = 0$ in equation (6), it can be shown that $c_1 = -c_2 = \varphi_0\sqrt{2}/2$. Then equation (6) yields (see also figure 2)

$$\frac{\varphi(T_0)}{\varphi_0} = \frac{e^{\frac{\pi h}{2}} + e^{-\frac{\pi h}{2}}}{2} \approx 1 + \frac{\pi^2 h^2}{8} \tag{7}$$

In general, after p roll cycles the growth is

$$\frac{\varphi(pT_0)}{\varphi_0} = \frac{e^{\frac{p\pi h}{2}} + e^{-\frac{p\pi h}{2}}}{2} \tag{8}$$

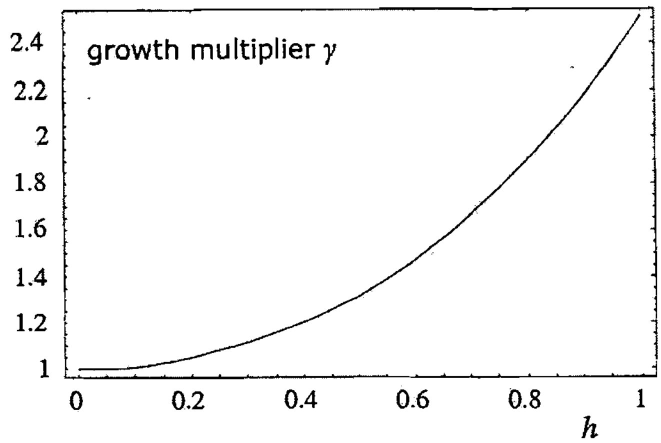

Figure 2. Growth of roll in the first region, for a single roll cycle, as function of the parametric amplitude h (no damping).

4. THE EFFECT OF DAMPING

Let us consider now a roll model based on Mathieu's equation with damping,

$$\ddot{\varphi} + 2k\dot{\varphi} + \omega_0^2\left[1 - h\cos(\omega_e t)\right]\varphi = 0 \tag{9}$$

k is half the dimensional damping divided by the roll moment of inertia including the added moment. The above can be transformed into an equivalent Mathieu equation (with no "explicit" damping term) by introducing the change of variable $\varphi = we^{-kt}$

$$\ddot{w} + \left(\omega_0^2 - k^2\right)\left(1 - \frac{\omega_0^2}{\omega_0^2 - k^2}h\ \cos\omega_e t\right)w = 0 \tag{10}$$

The effect of damping on the growth of amplitude is easily perceived. The combination of equations (8) and (10) yields the amplitude after one roll cycle

$$\varphi(T_0) = w(T_0)e^{-\frac{2\pi k}{\omega_0}} \approx \underbrace{\frac{e^{\frac{\pi h}{2}} + e^{-\frac{\pi h}{2}}}{2}}_{\gamma}\underbrace{e^{-\frac{2\pi k}{\omega_0}}}_{\gamma_d} w_0 = \frac{e^{\frac{\pi}{2}\left(h - \frac{4k}{\omega_0}\right)} + e^{-\frac{\pi}{2}\left(h + \frac{4k}{\omega_0}\right)}}{2}\varphi_0 \tag{11}$$

From the exponential term $e^{\frac{\pi}{2}\left(h-\frac{4k}{\omega_0}\right)}$ we extract the well-known condition of stability (i.e. no growth)

$$h_{crit} = \frac{4k}{\omega_0} \tag{12}$$

It is observed that the 'apparent' damping of the system is intensified at low frequencies.

Damping shifts the first region of instability (and in fact also all subsequent regions) upwards; i.e. it incurs a stabilising effect on the upright state, rendering it insensitive to small or moderate amplitude fluctuations of restoring and thus, loosely speaking, to waves of small or moderate height, even if these arrive with the right tuning. As a matter of fact, proper selection of damping can lead to negligible probability of encountering critical wave groups (in terms of the combination of the amount of height exceeded and frequency tuning). This provides the key instrument for eliminating parametric rolling through design.

The boundary line of the region of principal resonance obtained, e.g. with the method of harmonic balance, is

$$h_{crit} = 2\sqrt{\frac{(a-1)^2}{a^2} + \frac{4k^2}{a\,\omega_0^2}} \tag{13}$$

If we set $a = 1$ we come to an alternative derivation of the stability criterion equation (12). Strictly speaking, there should be a slight shift of the vertex, from $a = 1$ to $1+2k^2/\omega_0^2$ if damping up to second order were kept; then the lowest point of the boundary curve should also be slightly modified, to $4k\sqrt{\omega_0^2+3k^2}/\left(\omega_0^2+2k^2\right)$.

The growth inside this region may be determined, to first approximation, from equation (4). The exponent μ indicates how deeply we lie inside the instability region; while the phase σ (ranging from 0 to $-\pi/2$) indicates the distance from the sides of the boundary ($\sigma = 0$ at the right boundary and $\sigma = -\pi/2$ at the left boundary).

In general, the growth of amplitude after p roll cycles at exact resonance ($a = 1$) is

$$\varphi(pT_0) \approx e^{-\frac{2p\pi k}{\omega_0}} \left(\frac{e^{\frac{p\pi h}{2}} + e^{-\frac{p\pi h}{2}}}{2}\right) \varphi_0 \tag{14a}$$

For the first one or two cycles the growth is first order for k (i.e. k influences growth) but second order for h

$$\varphi(pT_0) \approx \left(1 - \frac{2\pi pk}{\omega_0} + \frac{2\pi^2 p^2 k^2}{\omega_0^2}\right)\left(1 + \frac{\pi^2 p^2 h^2}{8}\right)\varphi_0 \tag{14b}$$

A q-fold increase in roll amplitude from some initial angle of disturbance should entail, according to equations (14a) and (14b), p roll cycles

$$\ln q \approx -\frac{2p\pi k}{\omega_0} + \ln\left(\frac{e^{\frac{p\pi h}{2}} + e^{-\frac{p\pi h}{2}}}{2}\right) \tag{15}$$

Given that, after a few roll cycles the dominant exponential term of (15) is the one with positive sign, the above may be written further, approximately, as

$$\ln q \approx -\frac{2p\pi k}{\omega_0} + \frac{p\pi h}{2} - \ln 2 = \frac{p\pi}{2}\left(h - \frac{4k}{\omega_0}\right) - 0.693 \tag{16}$$

It follows that for a q-fold increase of amplitude the necessary number of cycles p should be

$$h - \frac{4k}{\omega_0} = \frac{0.693 + \ln q}{1.571\,p} \tag{17}$$

The time t_m required for this should be p times the natural period T_0

$$t_q = pT_0 \tag{18}$$

To demonstrate the usefulness of formula (17), let us think in terms of the following tentative criterion: a 10-fold increase of roll amplitude should never come about in less than 4 roll cycles for a wave with $\lambda/L = 1.0$, $H/\lambda = 1/20$. At principal resonance 4 roll cycles mean 8 critical wave encounters (it is possible to link this with the probability of encounter of a dangerous wave group with these dominant characteristics). The above translate into the following relationship for the parameters h, k

$$h - \frac{4k}{\omega_0} \le \frac{0.693 + \ln 10}{1.57 \times 4} \approx \frac{3}{6.282} = 0.477 \tag{19}$$

Criterion (17) targets transient response. Compared to equation (12), which is a criterion of asymptotic stability, it is superior in the sense that it does not suffer from the unrealistic assumption of the encounter of a critical wave group with infinite run length. The condition of asymptotic stability is recovered from equation (17) if we set $p \to \infty$ at the right hand side of (17). The transient response criterion (17) should probably be supplemented by a time requirement based on equation (18). For example, the 4 roll cycles would take 25.7 × 4 = 102.8 s. For very low natural frequencies, i.e. following seas, the required time becomes excessive. This leaves time for reaction (i.e. change of speed or heading) as soon as the beginning of the phenomenon is realised.

5. THE HIGHER REGIONS OF INSTABILITY

Among all parametric resonances, the principal one requires the lowest amplitude h (in general the required amplitude h is proportional to $k^{1/n}$ where n is the order of the resonance). Although a criterion like equation (12) is stringent, it informs about the minimal h required for reaching the region of principal resonance (n = 2). In some cases the ship's natural frequency is such that the scenario of principal resonance turns improbable, whereas the second resonance falls within the attainable speed range.

The perturbation analysis result for the boundary of the region of fundamental resonance is [Hayashi 1985]

$$h_{2+} = 4\sqrt{\frac{3}{5}}\,\frac{\sqrt{a-4}}{a}$$

$$h_{2-} = 4\sqrt{3}\,\frac{\sqrt{4-a}}{a} \tag{20}$$

The third or higher instability regions require long encounter periods [$\omega_e = (2/n)\omega_0 < \omega_0$ since $n = 3, 4 \ldots$], i.e. following seas and speeds quite near to wave celerity. However, these resonances have extremely low probability of occurrence, primarily due to the stabilising effect of damping.

There are several general formulae for predicting, approximately, the critical h for any resonance region, with damping taken into account. Two of these are Taylor & Narendra [1969]

$$h = \pi \frac{k}{\omega_0}\sqrt{a} \tag{21}$$

and Gunderson, Rigas & Van Vleck [1974]

$$h = \left(1 - \frac{k^2}{\omega_0^2}\right)\tanh\left(\pi \frac{k}{\omega_0}\sqrt{a}\right) \tag{22}$$

Equation (22) is free from the assumption of small damping that usually limits the applicability of perturbation methods.

Perhaps the most accurate expression for the minimal h is from Turyn [1993]

$$h = \frac{8}{n^2}\sqrt[n]{\frac{k\,n!}{2\omega_0}} \tag{23}$$

The above is calculated at $a = 1$ for $n = 1$ while for n $n \geq 2$ it should be calculated at

$$a = n^2 + \frac{8}{n^2 - 1}\sqrt[2/n]{\frac{k\,(n!)^2}{2\omega_0}} + \frac{k^2 n^2}{\omega_0^2} \tag{24}$$

It turns out that the required h (which is, practically, the representative of wave steepness through the "filter" of the hull) increases quickly from n = 1 to n = 2. Take for example a ship with $\zeta = k/\omega_0 = 0.06$. At principal resonance the threshold is $h = 4\zeta$ = 24% while at the fundamental it becomes, according to Turyn's formula, about 45.2%. Later resonances require even higher h although the rate of increase is lowered. The combination of requirements for high speed [see equation (2) and figure 1] and extreme restoring fluctuation (which calls for unrealistic wave dimensions), render the resonances above n=2 of almost negligible probability.

6. STOCHASTICALLY VARYING METACENTRIC HEIGHT: REVIEW OF STABILITY CRITERIA

For a section of the Market the occurrence of parametric rolling in a realistic sea is rather unfounded (see for example Ractliffe [2002]). Nonetheless, since the 70s and the 80s the problem has been recognised as important in naval architecture and a theoretical basis for the analysis of probabilistic parametric rolling has been developed [Price 1974; Vinje 1975; Skomedal 1982; Muhuri 1980; Roberts 1982; Dunwoody 1989]. Conceptually, all these approaches represent adaptations of theories from the field of stochastic differential equations (see for example Caughey & Dienes [1962]; Kozin [1969]; Arnold [1973]; Ariaratnam & Tam [1979]; Roberts & Spanos [1986]; Ibrahim [1986]). In the literature of mechanics it is more common to come across investigations of systems excited parametrically by white noise. However, in a parametric rolling investigation, the shape of the spectrum, and especially the "narrow-band" characteristic that creates a higher probability of encounter of wave groups, should not be neglected. As is well known, even for a linear stochastic system one could think of different definitions of stability. We may think in terms of convergence in probability, convergence in the mean and "almost sure" convergence. Hence a variety of criteria are found in the literature. Two concepts seem to be the most popular: The so called "almost sure" or "sample function" stability which implies that, as time tends to infinity, all samples, except for a set of measure zero, tend to the stationary solution; and the stability of moments. These criteria can take into account the damping and some characteristic of the spectrum of the fluctuating metacentric height.

For systems whose evolution is a diffusion Markov process the method of stochastic Lyapounov functions is one of the possible ways for deriving a criterion of almost sure asymptotic stability (see for example Gray [1967]). Assuming Gaussian fluctuation of GM with zero mean, Vinje [1975] derived the following simple ship stability criterion of parametric rolling based on a Lyapounov function $V(\varphi)$ that, as time goes to infinity, tends to 0 with probability 1.

$$\sigma < \frac{\sqrt{2\pi}\,k}{\omega_0} \tag{25}$$

For a harmonic process with amplitude h, the standard deviation is $\sigma = \frac{\sqrt{2}}{2}h$ and the above should become

$h < \dfrac{2\sqrt{\pi}\,k}{\omega_0}$ which falls midway between $\dfrac{4k}{\omega_0}$ of equation (12) and Taylor & Narendra's $\dfrac{\pi k}{\omega_0}$ from equation (21), i.e. the criterion may be too stringent.

By using the extreme properties of quadratic forms, Infante [1968] has shown that for a typical parametric system

$$\frac{d\mathbf{x}}{dt} = [\mathbf{A} + \mathbf{F}(t)]\mathbf{x} \tag{26}$$

a sample time evolution can be expressed as

$$\|\mathbf{x}(t)\| < \|\mathbf{x}(0)\|\, e^{\left[\frac{1}{t}\int_0^t \lambda(\tau)d\tau\right]t} \tag{27}$$

where, $\lambda(\tau)$ is the maximum eigenvalue of the matrix $\left\{ [\mathbf{A} + \mathbf{F}(t)]^T \mathbf{B} + \mathbf{B}[\mathbf{A} + \mathbf{F}(t)] \right\}\mathbf{B}^{-1}$. The constant matrix B is symmetric positive definite and determines the norm of the process x which is expressed as $\sqrt{\mathbf{x}^T \mathbf{B}\mathbf{x}}$. Assuming that $\mathbf{F}(t)$ is ergodic, then

$$\lim \frac{1}{t}\int_0^t \lambda(\tau)d\tau = \mathrm{E}[\lambda(\tau)] \quad \text{with probability 1.0} \tag{28}$$

It accrues that it suffices to request the expectation $\mathrm{E}[\lambda(\tau)] < 0$. The derived condition is a sharper one compared to equation (25)

$$\sigma < \frac{2k}{\omega_0} \tag{29}$$

"Almost sure" stability depends on the exponential growth rate of the response of the random system. This is critically influenced by the sign of the maximum Lyapounov exponent which may be regarded as the stochastic analogue of the real part of the largest eigenvalue of a linearised system under deterministic excitation.

$$\lambda = \lim \frac{1}{t} \ln\|\mathbf{x}(t; \mathbf{x}_0)\| \tag{30}$$

Here, by $\| \ \|$ is meant the norm of the stochastic process $\mathbf{X}$. A j-dimensional system should have j Lyapounov exponents at most. If the largest one is less than 0, then "almost surely" $\mathbf{x}(t; \mathbf{x}_0)$ should tend to 0 as time goes to infinity. The maximum Lyapounov exponent may be obtained from the so-called "Fustenberg-Khasminski" formula [Namachchivaya & Ramakrishnan, 2003]

$$\lambda = -\varepsilon^2\, 2k\left[-\frac{1}{2} + \frac{\pi\,\omega_0^2}{8k} S_h(2\,\omega_0)\right] \tag{31}$$

and the following definitions apply.

$$S_h(2\omega_0) = 2 \times \frac{1}{2\pi}\int_0^\infty R_{hh}(\tau)\cos(2\omega_0\tau)\,d\tau, \qquad R_{hh}(\tau) = E[h_t\, h_{t+\tau}] \tag{32}$$

To ensure that λ is negative, the condition is

$$k > \frac{\pi\omega_0^2}{4} S_h(2\omega_0) \tag{33}$$

The largest Lyapounov exponent does not provide information about the rate of convergence or about the stability of moments. This could be obtained from the moment Lyapounov exponent [Nolan & Namachchivaya 1999] that, however, leads to criteria of higher stringency. To determine the p-th moment stability we solve for the p-th moment of the amplitude response and compute the moment Lyapounov exponent defined as

$$\lambda^{(p)}(\varphi_0) = \lim \frac{1}{t} \ln \mathrm{E}\|\varphi(t;\, \varphi_0)\|^p \tag{34}$$

where $\mathbf{E}$ denotes the expectation. As previously, if $\lambda^{(p)}(\varphi_0) < 0$ then $\mathrm{E}\|\varphi(t;\, \varphi_0)\|^p \to 0$ as time tends to infinity (condition of p-th moment stability). For our parametric system the above condition produces the following moment stability criterion

$$\lambda^{(p)} = \varepsilon^2\, 2pk\left[-\frac{1}{2} + \frac{p+2}{16}\frac{\pi\,\omega_0^2}{k} S(2\,\omega_0)\right] \tag{35}$$

The condition ensuring negative $\lambda^{(p)}$ is

$$k > \frac{p+2}{8}\pi\omega_0^2\, S_h(2\omega_0) \tag{36}$$

Roberts [1982] applied to roll stability the method of stochastic averaging of Stratonovitch (see Roberts & Spanos [1986] for a review) in order to circumvent the problem of dealing with non-white noise parametric excitation. He proposed that, since the joint process of roll amplitude and phase (A, ϑ) converges weakly (as the damping goes to zero) to a two-dimensional Markov process, (A, ϑ) is governed by the so-called 'Ito equations' (see, for example, Ito [1951]) from which a Fokker-Planck-Kolmogorov (FPK) equation is derived for the transition probability density of the joint process. Especially in the amplitude Ito equation the phase is not present and thus $A(t)$ can be treated as a one-dimensional Markov process whose transition probability density should depend only on $A(t)$. The assumption of the stationary nature of $A(t)$ means that the time derivative of the transition probability function should be zero. This simplifies the calculations resulting in the criterion of Ariaratnam and Tam [1979] for sample function stability, which involves the natural frequency, the damping and the spectral density of the fluctuating *GM* process at twice the natural frequency

$$k > \frac{\pi \omega_0^2 S_h(2\omega_0)}{4} \tag{37}$$

The above criterion, which is useful for linear stability (i.e. only for the upright state), is identical with criterion (29) based on the largest Lyapounov exponent. But, as noted also by Bulian *et al.* [2003], it is not useful for telling the characteristics of the ensuing nonlinear response because in the expression of roll amplitude the nonlinear stiffness term does not partake in the averaging process. Skomedal [1982] carried out a comparison against model test data of Roberts' predictions of the variance of roll amplitude and found that the predicted roll variance is overestimated.

Stability of sample functions is perhaps what interests us most. The criteria for the stability of moments are more severe in terms of the required minimal damping; by 50% (in terms of minimal k) for the first moment and by 100% for the second.

Dunwoody [1989] proposed a sample function stability criterion on the basis of the observation (already apparent from our investigations in previous sections) that the fluctuations of GM produce an effect that works like a reduction of roll damping. By requesting that the damping ratio is greater than the expectation of this effective damping due to GM's fluctuation, a criterion of stability can be deduced which is identical with equation (12). Assuming that the amplitude h is Rayleigh distributed, its expectation is

$$E[h] = \sqrt{\frac{\pi}{2}}\sigma_h < \frac{4k}{\omega_0} \tag{38}$$

7. A UNIFYING APPROACH BASED ON WAVE GROUPS

It is well known that higher waves tend to arise in groups. As the nearly regular characteristics of waves in a group are essential for giving rise to resonant motions like parametric rolling, there is a meaningful link between the probabilistic nature of ocean waves and the deterministic analysis of the earlier sections of this paper. The probability of occurrence of parametric rolling could be assumed to be equal to the probability of encountering a wave group with a suitable run length and exceeding the threshold wave height determined from the deterministic analysis, given that the frequency falls in the critical range. This viewpoint is indeed fundamentally different form the conventional one of linear seakeeping analysis where the wave field is approached as the superposition of regular waves with arbitrary phase and energy. Instead, here the extreme wave field is approached as a sequence of wave groups [Spyrou 2004].

Attention to wave groups is not completely new. Assuming no correlation between successive wave heights and without setting any requirement about the period, Blocki [1980] determined the probability of encounter of a dangerous wave group by using a Rayleigh probability density function for the amplitude, whose integration from a critical level ρ to infinity should produce the probability of exceeding ρ. The probability of encountering a succession of j waves (i.e. a group) having this property should be calculated from the well known formula (e.g. Goda [1976])

$$P(A_j) = p^{j-1}(1-p) \tag{39}$$

Takaishi *et al.* [2000] targeted the probability of encountering a "high run" of waves and developed an operational guidance for shipmasters. His approach was based on Longuet-Higgins' [1984] statistical properties of wave groups. The key point was the observation that, in following/quartering seas, there is a range of speeds where all the energy of the wave field is concentrated within a very narrow range of encounter wave frequencies due to the Doppler effect. In other words, the encounter spectrum becomes very narrow, which increases the probability of encounter of a "high run". The well-known relation between the encountered wave spectrum $S(\omega_e, \mu)$ and the "true" wave spectrum is

$$S(\omega_e, \mu) = \frac{S(\omega)}{\left|1 - \frac{2\omega U \cos\mu}{g}\right|}, \quad \omega_e = \omega - \frac{\omega^2 U}{g}\cos\mu \tag{40}$$

where μ is the encounter angle and U is ship speed. As a resonance mechanism, parametric rolling entails the encountered wave frequency to be in some approximate relationship with the natural roll frequency. If, at the encounter frequencies of energy concentration, this approximate relationship holds true, then a very dangerous setup for parametric rolling is in place. However, from a design point of view the anticipated speed U in a storm is quite uncertain and some probability distribution $P(U)$ should be assumed instead of a discrete value.

An improved approach concerning the same problem could incorporate theoretical or parametric models for the joint distributions of wave parameters because wave height and period, successive wave heights, as well as successive wave periods of extreme waves are generally correlated. Unfortunately, the desirable multivariate distributions are not yet available in the literature. For practicality, one has to make certain assumptions about the correlation of key parameters in a wave group and opt to use available bivariate distributions. Bivariate probability density functions of height and period of ocean waves have been proposed by a number of investigators, (see for example Longuet-Higgins [1975], Cavanié *et al.* [1976], Longuet-Higgins [1983]). The last one of Longuet-Higgins is shown below as described by Demirbilek & Linwood Vincent [2002]

$$p(H,T) = \frac{\pi f(\nu)}{4}\left(\frac{H_*}{T_*}\right)^2 e^{-\frac{\pi H_*^2}{4}\left(1 + \frac{1 - \frac{\sqrt{1+\nu^2}}{T_*}}{\nu^2}\right)} \tag{41}$$

where

$$H_* = \frac{H}{\overline{H}} \tag{42}$$

$$T_* = \frac{T}{\overline{T}_z}, \tag{43}$$

$$f(\nu) \approx \frac{2\left(1+\nu^2\right)}{\nu + \frac{\nu}{\sqrt{1+\nu^2}}} \tag{44}$$

$$\nu = \sqrt{\frac{m_0 m_2 - m_1^2}{m_1^2}} \tag{45}$$

As usual m_0, m_1, m_2, are respectively zeroth, first and second moment of the (encountered) wave spectrum, $\overline{H}$ is the mean wave height; and $\overline{T}_z$ is the mean zero- upcrossing period that is calculated from the spectral moments, $\overline{T}_z = 2\pi\sqrt{m_0/m_2}$. It is to be noted that the spectral width parameter ν, and subsequently the distribution, depends only on the first three moments.

The probability of encounter of a wave with height above the critical one H_c and with ω_e near to $2\omega_0$ (say ±20%), should be calculated from the double integral

$$P\left[H > H_c,\ 0.8T_c < T_e < 1.2T_c\right] = \int_{H_c}^{\infty} \int_{0.8T_c}^{1.2T_c} p(H, T_e)\, dT_e\, dH \tag{46}$$

Thereafter, the probability of encountering a wave group with these characteristics and, in addition, a run length j, should be given by again applying equation (39). Even this approach, however, takes into account neither the correlation between successive wave periods nor the correlation between successive wave heights.

Distributions of successive wave periods have been employed by Myrhaug *et al.* [2000] for developing a probabilistic assessment of beam-sea rolling. Although ideally this correlation should be inside a probabilistic assessment of parametric rolling, it is less important compared to the correlation of successive wave heights because, due to the interference of speed, the critical wave period is relatively uncertain at the design stage. On the other hand, parametric instability becomes possible only if a certain wave height is exceeded. As a matter of fact, if a choice has to be made, the correlation of wave heights should be higher priority.

The degree of correlation between successive wave heights depends on the sharpness of the spectral peak. For the effect of the spectral bandwidth on the distribution of wave height see, for example, Kimura [1980], Tayfun [1983] and Longuet-Higgins [1984]. Stansell *et al.* [2002] found that, as bandwidth increases, there is a rather slight reduction in the mean run and group length, up to a bandwidth $\nu = 0.6$ beyond which they become rather insensitive. To obtain a sense of magnitude we note that $\nu = 0.425$ for a Pierson Moskowitz and $\nu = 0.389$ for a JONSWAP spectrum).

According to Tayfun, the sharpness of the spectral peak reflects the variability of height between successive waves and is best represented by the correlation coefficient of the wave envelope R_{HH} which could be calculated as

$$R_{HH} = \frac{E(\kappa) - \left(1-\kappa^2\right)\frac{K(\kappa)}{2} - \frac{\pi}{4}}{1 - \frac{\pi}{4}} \approx \frac{\pi}{16 - 4\pi}\left(\kappa^2 + \frac{\kappa^4}{16} + \frac{\kappa^6}{64}\right) \tag{47}$$

$E(\ )$, $K(\ )$ are complete elliptic integrals of the first and second kind, respectively. The correlation parameter κ could be calculated as follows (see Stansell *et al.* [2002] for an extensive discussion on alternative methods).

$$\kappa(\overline{T}_z) = \frac{1}{m_0}\sqrt{A^2 + B^2}, \tag{48}$$

$$A = \int_0^{\infty} S(f) \cos 2\pi f \overline{T}_z\, df, \tag{49}$$

$$B = \int_0^{\infty} S(f) \sin 2\pi f \overline{T}_z\, df \tag{50}$$

where $f = \omega_e/2\pi$ (Hz) and $S(\ )$ is the encounter spectrum. Goda [1976] has found that, for swells, the correlation coefficient R_{HH} is about 0.6 while for wind waves it is only about 0.2.

Assuming that successive wave heights follow a Rayleigh distribution, Kimura [1980] derived the following bivariate probability density function $p(H_1, H_2)$ for consecutive wave heights

$$p_{HH}(H_1, H_2) = \frac{4H_1 H_2}{\left(1-\kappa^2\right)H_{rms}^4} e^{-\frac{\left(H_1^2+H_2^2\right)}{\left(1-\kappa^2\right)H_{rms}^2}} I_0\left(-\frac{2\kappa H_1 H_2}{\left(1-\kappa^2\right)H_{rms}^2}\right) \tag{51}$$

where H_{rms} is the root mean square wave height and I_0 is the modified Bessel function of zeroth order. The probability of having two consecutive wave heights above the critical height H_c will then be

$$P\left(H_{i+1} \geq H_c \middle| H_i \geq H_c\right) = \frac{\int_0^{H_c}\int_0^{H_c} p_{HH}(H_1, H_2)\, dH_1\, dH_2}{\int_0^{H_c} p_H(H)\, dH} \tag{52}$$

where $p_H(H)$ is the marginal probability density which is Rayleigh type

$$p_H(H) = \frac{2H}{H_{rms}^2} e^{-\frac{H^2}{H_{rms}^2}} \tag{53}$$

The assumption of a Markov chain for successive wave heights means that the probability of occurrence of a group with length j and heights above H_c can be calculated again on the basis of equation (39) where this time, however, $P = P\left(H_{i+1} \geq H_c \middle| H_i \geq H_c\right)$. To overcome the neglect of wave period, the above could be multiplied by a susceptibility factor indicating whether the speed range of the ship produces encounter frequencies that overlap with the frequencies of principal resonance.

8. RESPONSE FEATURES DUE TO GEOMETRICAL NONLINEARITIES

We shall revert now to the deterministic case, in order to consider behaviour away from the vicinity of the upright state. As is well known, there is no reason for the parametric growth of roll to persist up to infinity and thus lead by necessity to capsize. The detuning due to the nonlinear character of the *GZ* curve combined with the increased dissipation due to the mild nonlinearity of damping, creates the prospect of realising bounded rolling with moderate amplitude. In effect, for a typical parametric growth with nonlinear restoring the boundary curves of stability discussed earlier represent loci of sub-critical and supercritical bifurcations creating, respectively, unstable and stable oscillatory behaviour (see for example Skalak & Yarymovych [1960], Soliman & Thompson [1992]). At a supercritical bifurcation the new type of stable behaviour emerges smoothly while at a sub-critical the new type comes about with a jump.

A truly interesting observation is that the instability boundary curves determined earlier for the upright state do not entirely contain the domain where parametric oscillations are realisable. At first sight, in an idealised environment of a periodic seaway that is free from other external disturbances, the system should find no reason to leave the upright state as long as the combination of frequency ratio and parametric amplitude corresponds to some point in the region of stability. Nonetheless, the emerging stable roll oscillations need not be confined inside the "tongues" of the linear system and stable oscillations also exist well outside these regions [Scalak & Yarymovich 1961; Thompson & Soliman 1993; Francescutto & Dessi 2001]. Should the stable upright condition be sufficiently disturbed, this oscillatory behaviour can be incurred in an abrupt way. A well-focused experimental effort is required for establishing this theoretical prediction.

To take these points further, let us consider a Mathieu-type roll equation with a nonlinear term

$$\ddot{\varphi} + 2k\dot{\varphi} + \omega_0^2\left[1 - h\cos(\omega_e t)\right]\varphi - c_3\omega_0^2\varphi^3 = 0 \qquad (54)$$

The constant c_3 could be negative, in which case we are practically confined to studying the oscillations corresponding to the initial part of the $GZ(\varphi)$ curve which may be of "hardening spring" type; or it could be positive in which case we may be referring generically to the whole $GZ(\varphi)$ curve up to the angle of vanishing stability. In the last case, the vanishing angle is linked to c_3 with the relationship $\varphi_v = 1/\sqrt{c_3}$. There is no strict justification for assuming c_3 to be non-time-dependent. If the oscillations were of reasonably small amplitude, the product of the 3rd power of the scaled roll angle times the amplitude (assumed as small by necessity) of the fluctuating part of nonlinear stiffness might be considered negligible. Hence a representation like equation (49) could be taken as the basic generic model.

With the definitions

$$\tau = \omega_e t, \qquad a = 4\frac{\omega_0^2}{\omega_e^2} \qquad (55)$$

the above can be written further as

$$\frac{d^2\varphi}{d\tau^2} + \frac{2k\sqrt{a}}{\omega_0}\frac{d\varphi}{d\tau} + a(1 - h\cos 2\tau)\varphi - a c_3\varphi^3 = 0 \qquad (56)$$

If we confine ourselves to symmetric-type responses (it sounds strange that non-symmetric responses could exist; but this is well-known for several parametrically-excited mechanical systems), we may write the solution of the above as a Fourier series with odd terms only as

$$\varphi = \sum_{\nu=1}^{\infty}\left[A_\nu(\tau)\sin\nu\tau + B_\nu(\tau)\cos\nu\tau\right], \quad \nu = 1, 3, 5, \ldots \qquad (57)$$

In general, the first term ($\nu = 1$) in the series suffices for the level of accuracy sought by the present analysis. Substitution of the approximate solution $\varphi = A_1(\tau)\sin\tau + B_1(\tau)\cos\tau$ into (57) and assuming that the sine and cosine amplitudes $A_1(\tau)$, $B_1(\tau)$, respectively vary slowly in time, yields the following two equations.

$$\begin{aligned}\dot{B}_1(\tau) = &\left(-1 + a + \frac{ah}{2}\right)A_1(\tau)\\ &-\frac{3}{4}ac_3 A_1(\tau)\left[A_1^2(\tau) + B_1^2(\tau)\right]\\ &-\frac{2k\sqrt{a}}{\omega_0}B_1(\tau)\end{aligned} \qquad (58)$$

$$\begin{aligned}\dot{A}_1(\tau) = &-\frac{2k\sqrt{a}}{\omega_0}A_1(\tau)\\ &+\frac{3}{4}ac_3 B_1(\tau)\left[A_1^2(\tau) + B_1^2(\tau)\right]\\ &-\left(-1 + a - \frac{ah}{2}\right)B_1(\tau)\end{aligned} \qquad (59)$$

At steady state there is no change of amplitude and hence we should request

$$\left(-1 + a + \frac{ah}{2}\right)A_1 - \frac{3}{4}ac_3 A_1\left(A_1^2 + B_1^2\right) - \frac{2k\sqrt{a}}{\omega_0}B_1 = 0 \qquad (60)$$

$$\frac{2k\sqrt{a}}{\omega_0}A_1 - \frac{3}{4}ac_3 B_1\left(A_1^2 + B_1^2\right) + \left(-1 + a - \frac{ah}{2}\right)B_1 = 0 \qquad (61)$$

By expressing A_1,B_1 in terms of the steady roll amplitude $A = \sqrt{A_1^2 + B_1^2}$ and phase ϑ, i.e. $A_1 = A\sin\vartheta$, $B_1 = A\cos\vartheta$, and following some further algebraic manipulation we arrive at the following set

$$\sin 2\vartheta = \frac{4k}{h\omega_0\sqrt{a}} \quad (62a)$$

$$\cos 2\vartheta = \frac{2}{h}\left(1 - \frac{1}{a} - \frac{3}{4}c_3A^2\right) \quad (62b)$$

The right hand side of (62a) is always positive; hence $0 \le \vartheta \le \pi/2$. The right hand side of the lower expression may be positive or negative, depending on the sign of the coefficient of the nonlinear term c_3 and on whether we lie to the left or to the right of $a = 1$.

The combination of the above produces the following explicit formula for the amplitude.

$$A^2 = \frac{4}{3c_3}\left[\left(1 - \frac{1}{a}\right) \mp \sqrt{\frac{h^2}{4} - \frac{4k^2}{a\omega_0^2}}\right] \quad (63)$$

Setting $A \to 0$ we find the curve whereon the oscillations are created. It comes as no surprise that this curve is independent of the nonlinear coefficient n and it coincides with the boundary of linear stability. Also, the term inside the square root, as well as the whole expression of A^2, should be non-negative. For, say, an initially hardening restoring ($c_3 < 0$) these yield

$$h \ge \frac{4k}{\omega_0\sqrt{a}} \quad \text{and} \quad a \le 1 \quad (64)$$

Essentially, equations (64) define a locus of "saddle-node" bifurcations where the response curve is "folded". The unstable periodic orbits that emerged at the left boundary of the region of instability go through a U-turn and they are rendered stable. Equation (64) determines the true boundary of periodic response. For a certain level of h, the region with oscillations is wider than the one predicted from linear analysis. The stability boundary of the principal resonance is in fact a bifurcation locus as shown in figure 3. For hardening restoring the right part (a>1) is of super-critical type and the left part (a>1) of sub-critical type. These properties are reversed when the restoring is softening.

For $c_3 < 0$ ("hardening *GZ*") the amplitudes are

$$a \ge 1: \quad A_{a=1+} = \sqrt{-\frac{4}{3c_3}}\sqrt{\left(\frac{1}{a} - 1\right) + \sqrt{\frac{h^2}{4} - \frac{4k^2}{a\omega_0^2}}} \quad (65a)$$

$$a < 1: \quad A_{a=1-} = \sqrt{-\frac{4}{3c_3}}\sqrt{\left(\frac{1}{a} - 1\right) \pm \sqrt{\frac{h^2}{4} - \frac{4k^2}{a\omega_0^2}}} \quad (65b)$$

Figure. 4 is a bifurcation diagram, i.e. it shows the change of steady-state roll response as a control parameter, in this case h, is varied. Even without carrying out a formal stability analysis, to an experienced eye the stability of the emerging steady roll oscillations is quite obvious. On the right boundary (higher a) of the instability region a supercritical bifurcation takes place, whereas the boundary at a<1 gives birth to a sub-critical one (see also figure 3).

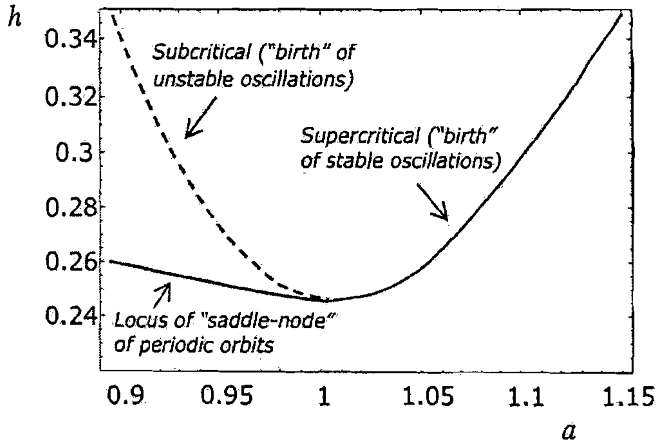

Figure 3. Boundary of parametric roll: $k = 0.015s^1$, $\omega_0 = 0.2448s^{-1}$ (hardening restoring).

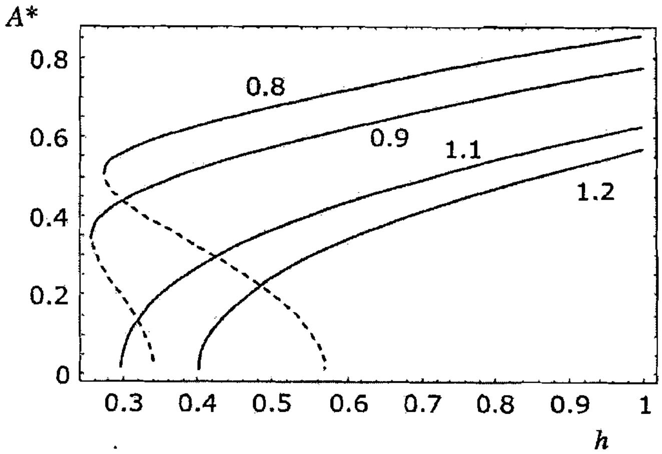

Figure 4. Amplitude of response for hardening restoring
$(A^* = \sqrt{\frac{3c_3}{4}}A = 0.866\frac{A}{\varphi_v})$.

The domain of oscillatory behaviour can easily be found with some manipulation of equation (51)

$$h = \sqrt{4\left[\frac{3c_3}{4}A^2 - \left(1 - \frac{1}{a}\right)\right]^2 + \frac{16k^2}{\omega_0^2 a}} \quad (66)$$

The combinations of (h,a) that give rise to oscillations of predefined A^* is shown in figure 5. It is easily proven that the descending part of each iso-A^* curve corresponds to stable rolling and the ascending to unstable rolling. The boundary of stable rolling is reconfirmed (thick continuous line). As we have multiple coexisting stable responses, the initial conditions and the availability of sufficiently strong external disturbances determine whether the ship can stay upright, or should adopt the one (desired) or the other (undesired and possibly dangerous) way of behaviour.

In figure 6 is shown the variation of the roll amplitude A for $c_3 = 2.35$ as a function of the linear damping and the amplitude of parametric forcing. As deduced from expression (63), the amplitude A goes with the square root of both the parametric forcing and the damping. For small (yet realistic for many operating ships) damping, the effect on the response amplitude is relatively small. This is perhaps counterintuitive, given that damping is the most critical parameter for the onset of parametric rolling in the first place. Whilst the same applies for h, the latter should be quite high (well above $4k/\omega_0$) since the occurrence of parametric rolling is taken here as a fact. A, say, 10% increase of h incurs a considerably larger quantitative effect on the amplitude than a 10% reduction of damping. Another influential parameter that is linked to ship geometry is the coefficient c_3 of cubic stiffness: on the basis of (63), c_3 is inversely proportional to A^2 (see also figure 7).

For $c_3 > 0$ (softening GZ) the amplitudes are

$$a \ge 1: \qquad A_{a=1+} = \sqrt{\frac{4}{3c_3}}\sqrt{\left(1-\frac{1}{a}\right)\mp\sqrt{\frac{h^2}{4}-\frac{4k^2}{a\omega_0^2}}} \qquad (67a)$$

It is noted above that there are two solutions corresponding to the stable and unstable part. Their maximum values (for "large" a) are

$$A_{a=1+} = \sqrt{\frac{4}{3c_3}}\sqrt{1\mp\frac{h}{2}} \qquad (67b)$$

However it should not be disregarded that the formula was derived for the vicinity of principal resonance and also that if the roll amplitude becomes large a more accurate representation of restoring is entailed.

$$a < 1: \qquad A_{a=1-} = \sqrt{\frac{4}{3c_3}}\sqrt{\left(1-\frac{1}{a}\right)+\sqrt{\frac{h^2}{4}-\frac{4k^2}{a\omega_0^2}}} \qquad (67c)$$

In figure 8 is shown the change of roll amplitude as the parametric forcing h is raised, for frequency ratios surrounding $a = 1$. Contours of iso-h on the plane of roll amplitude against a are shown in figure 9.

The bifurcation diagram of figure 8 suggests that, for a less than 1.0, the oscillations are stable as they emerge from a supercritical bifurcation. To the contrary, for a above 1.0 the oscillations are the result of a subcritical bifurcation, hence they are initially unstable (dashed line). However, these unstable oscillations later revert to stable at saddle-node bifurcation points. It is noted that the vanishing angle is approached quicker for the higher a (which for a given ship could be interpreted as a lower frequency of encounter) and at $a = 1.2$ the required h is more than 1.0. It should be recalled that the critical h for parametric rolling of the discussed containership is $4k/\omega_0 = 0.245$; i.e. the distance is substantial.

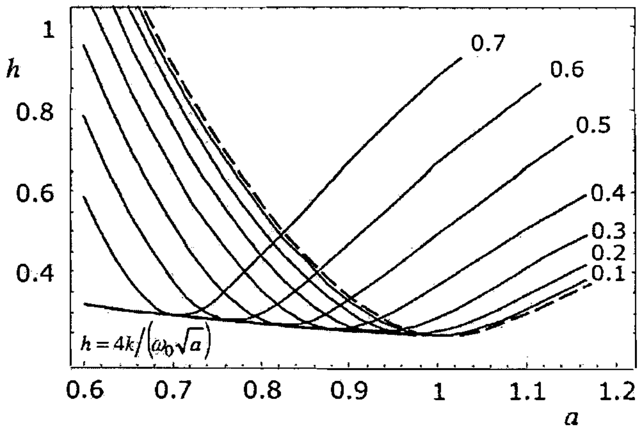

Figure 5. Iso-A curves (from 0.1 to 0.7 rad) for $k = 0.015s^{-1}$, $c_3 = 2.35$ (hardening), $\omega_0 = .2448s^{-1}$.

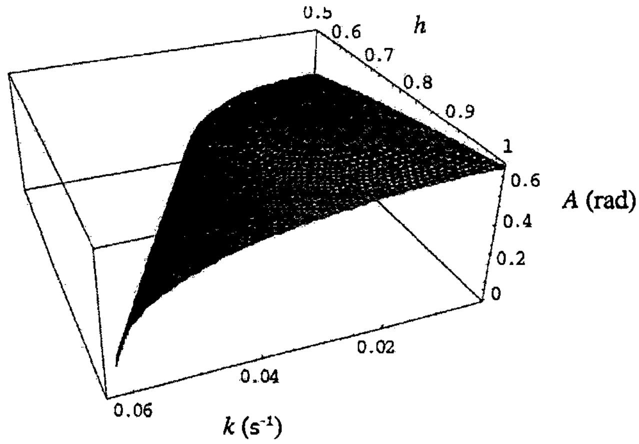

Figure 6. Steady response as function of k, h.

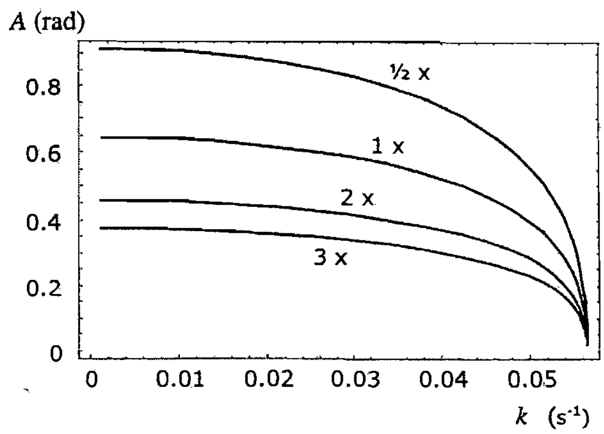

Figure 7. Effect of damping on the amplitude of periodic response with parameter the coefficient of nonlinear stiffness ("hardening").

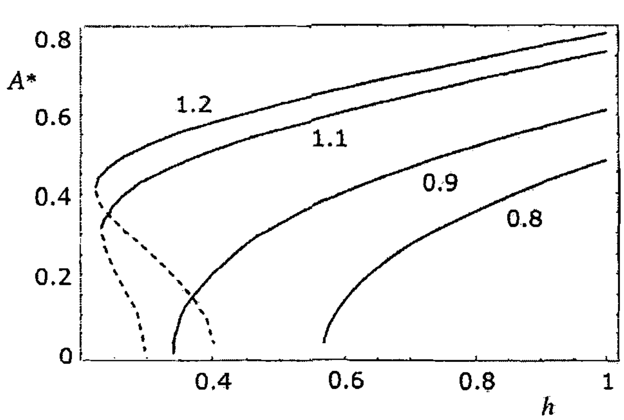

Figure 8. Amplitude of response for softening restoring, $k = 0.015\ \mathrm{s}^{-1}$.

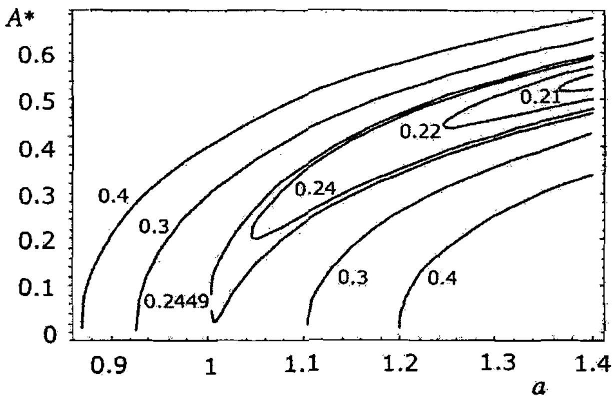

Figure 9. Contours of iso-h.

(a) Transient Behaviour

From the equations of amplitude given earlier and those of phase given below, we observe that the amplitude is changed indirectly through the variation of the phase

$$\frac{d}{d\tau}A^2(\tau) = A^2(\tau)\left(a\,h\,\sin 2\vartheta(\tau) - \frac{4k\sqrt{a}}{\omega_0}\right) \qquad (68a)$$

$$\frac{d\vartheta(\tau)}{d\tau} = \frac{h}{2}\cos 2\vartheta(\tau) - 1 + \frac{1}{a} + \frac{3}{4}c_3\,A^2(\tau) \qquad (68b)$$

An approximate expression for transient roll on the basis of the perturbation method of Krylov-Bogoliubov can be found in Blocki [1980].

(b) Effect of the Fifth Order Term (Iinitially Hardening, then Softening)

Consider again the roll equation with a fifth-order polynomial for restoring which can better take into account the detail of the GZ curve up to larger inclinations.

$$\frac{d^2\varphi}{d\tau^2} + \frac{2k\sqrt{a}}{\omega_0}\frac{d\varphi}{d\tau} + a(1 - h\cos 2\tau)\varphi - c_3\,a\varphi^3 - c_5\,a\varphi^5 = 0 \qquad (69)$$

The pair of equations that in this case have to be simultaneously zero become

$$\sin 2\vartheta = \frac{4k}{h\omega_0\sqrt{a}} \qquad (70)$$

$$\cos 2\vartheta = \frac{2}{h}\left(1 - \frac{1}{a} - \frac{3}{4}c_3\,A^2 - \underline{\frac{5}{8}c_5\,A^4}\right) \qquad (71)$$

We notice that only the lower one has been modified (by the underlined term). As expected, the boundary of parametric rolling remains unchanged and thus the criterion $h = 4k/\omega_0\sqrt{a}$ does not depend on the order of the restoring polynomial. The amplitude A is obtained from the quadratic equation

$$\frac{5}{8}c_5\,A^4 + \frac{3}{4}c_3\,A^2 - \left(1 - \frac{1}{a}\right) \pm \sqrt{\frac{h^2}{4} - \frac{4k^2}{\omega_0^2 a}} = 0 \qquad (72)$$

and the explicit expression of A is

$$A^2 = -\frac{3\,c_3}{5\,c_5} \pm \sqrt{\left(\frac{3\,c_3}{5\,c_5}\right)^2 - \frac{8}{c_5}\left(-1 + \frac{1}{a} \pm \sqrt{\frac{h^2}{4} - \frac{4k^2}{\omega_0^2 a}}\right)} \qquad (73)$$

We select a GZ curve (see figure 10) very close to that of the post-panamax containership discussed by France *et al.* [2001]. The selected values for the coefficients c_3 and c_5 are, respectively, -0.14 and 0.25. The amplitudes, as functions of the parametric term h, for the frequency ratios examined earlier, are shown in figure 11. Several changes of stability are taking place on each one of these curves. An interesting feature of this diagram is that it shows the behaviour at large angles where the fifth order term of the restoring function becomes influential. Contrasting figure 11 with figure 4 is enlightening in this respect.

A supplementary criterion based on steady amplitude may be introduced: for a steep wave, say with $\lambda/L = 1.0$, $H/\lambda = 1/20$, the max roll amplitude should not exceed, say, 15 deg (the same angle is proposed also in the ABS guide of parametric roll). From equation (73) is obtained the combination of restoring and damping characteristics that guarantee this limiting angle is not exceeded

(c) Nonlinear Damping

By including a cubic damping term $\delta\,\dot{\varphi}^3$ the equation of amplitude (67a) is modified directly; however that of phase is not.

$$\frac{d}{d\tau}A^2(\tau) = A^2(\tau)\left(a\,h\,\sin 2\vartheta(\tau) - \frac{4k\sqrt{a}}{\omega_0} - \frac{3\omega_0\delta}{2\sqrt{a}}\right) \qquad (74a)$$

$$\frac{d\vartheta(\tau)}{d\tau}=\frac{h}{2}\cos 2\vartheta(\tau)-1+\frac{1}{a}+\frac{3}{4}c_3 A^2(\tau) \qquad (74b)$$

Thereafter the steady amplitude should become

$$A^2=\frac{4}{3c_3}\left[\left(1-\frac{1}{a}\right)\mp\sqrt{\frac{h^2}{4}-\left(\frac{2k}{\omega_0\sqrt{a}}+\frac{3}{4}\frac{\omega_0\,\delta}{a^{3/2}}\right)^2}\right] \qquad (75)$$

The nonlinear damping coefficient is much smaller than the coefficient of the linear. Quantitative assessment of the various contributions to the amplitude A suggests that the effect of nonlinear damping to the reduction of A is much lower than that of the linear.

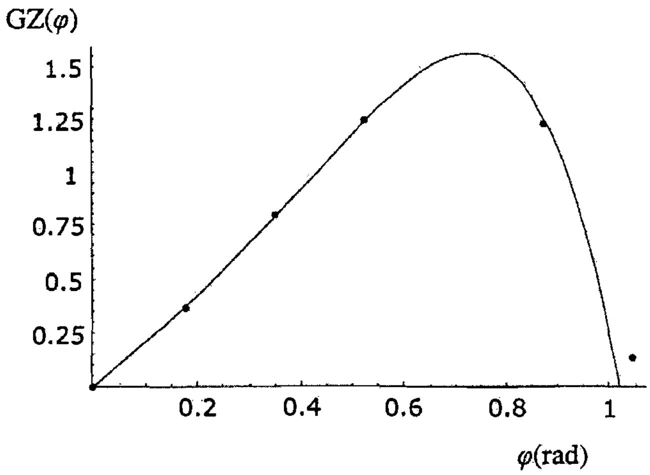

Figure 10. The exact (dots) and the polynomial fit (line) of the GZ curve.

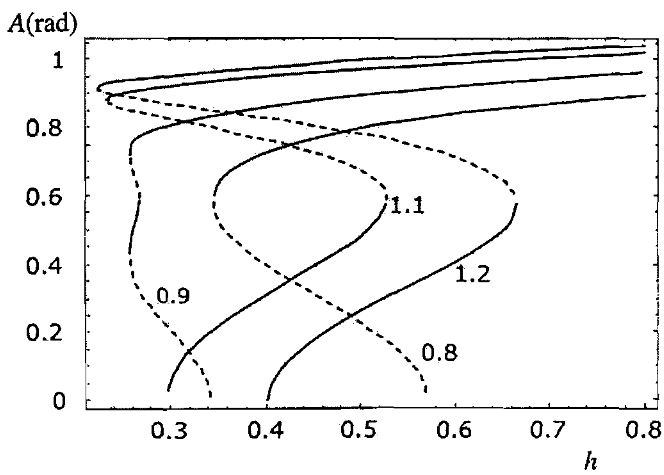

Figure 11. Amplitude of roll oscillation for 5th order restoring, assuming time dependence only in the linear term. The parameter is a.

(d) Capsize due to Parametric Rolling

Could there be an imminent danger of capsize if a ship is caught in parametric rolling? At a theoretical level this could be assessed by identifying the h level where loss of integrity of the basins of attraction of the emerging oscillatory roll responses is initiated. Relevant formulae are found, for example, in Kan [1992], Esparza & Falzarano [1993] and Nayfeh & Balachandran [1995] based on Melnikov's method. However, as the required h for loss of integrity is in general quite high (and notably, it is reduced at very low encounter frequencies), it seems that the danger of capsize is not high, unless the ship suffers from a short stability range combined with very significant GZ variations even in moderately steep waves.

9. COUPLING WITH PITCH AND HEAVE MOTIONS

In head waves, when the wave length becomes comparable to ship length, vertical ship motions often reach their maximum. Hence added resistance and loss of speed should be expected. The added resistance in regular head waves may be predicted by the well-known strip-theory formula of Gerritsma & Beukelman [1972]

$$R_W=\frac{k}{2\omega_e}\int_L\left[B_{33}(x)+U\frac{dA_{33}(x)}{dx}\right]w_{amp}^2(x)dx \qquad (76)$$

$A_{33}(x)\,B_{33}(x)$ are added mass and damping of a transverse section located at x along the ship, U is the ship's speed and $w_{amp}(x)$ is the amplitude of vertical velocity of the section relative to the wave. A coupled model of surge, heave and pitch should provide the mean speed for the specific ship-wave encounter scenario. Furthermore, pitch and heave should modify the instantaneous waterline as well as the location of the centre of buoyancy, resulting in an extra modulation of roll restoring with the pitch angle $\vartheta(t)$ and heave displacement $z(t)$. Therefore, if we intended to stretch out the conventional analysis of parametric rolling for application to a head sea scenario, the actual speed and restoring modulation should be determined in advance. However, a 4-degree coupled model of surge, heave, pitch and roll is the best option for a head sea scenario, not least because it can cope with a possible transfer of energy between the vertical modes and roll. A single roll model is limited in its capability to incorporate such a full dynamic effect.

If first order analysis is performed (where heave, pitch and wave slope are first order quantities), one could superimpose the GZ fluctuation due to pitch and heave to the semi-static one. The parametric driver of roll motion from concurrent heave and pitch could arise even in still water. Froude had observed this. Nearly 50 years ago, Paulling & Rosengerg [1959] showed that a prescribed harmonic heave or pitch motion (which in a different context could be assumed to be the response to direct wave excitation) acts like parametric excitation for roll. This should affect the predicted critical roll damping quantitatively but there should be no qualitative change concerning the character of the instability and the encounter frequency where it might occur because, in this semi-coupled scenario, pitch and heave are unaffected by the simultaneous rolling. However, the two other motions through second-order stiffness terms influence roll. More specifically, for small excursions from the upright condition the restoring function could be expressed as the following superposition

$$
\begin{aligned}
&\rho g \nabla_0 GM_0 (1 + h\cos\omega_e t)\varphi \\
&+\rho g \frac{\partial[\nabla(\zeta,\vartheta) GM(\zeta,\vartheta)]}{\partial\zeta}\zeta\varphi \\
&+\rho g \frac{\partial[\nabla(\zeta,\vartheta) GM(\zeta,\vartheta)]}{\partial\vartheta}\vartheta\varphi
\end{aligned}
\tag{77}
$$

The subscript 0 refers to the "still" waterline. The origin of our "right-handed" body-fixed axes is placed at the ship's centre of gravity with the x-axis pointing along the ship, y transversely and z vertically (positive downwards). We define heave displacement as $\zeta = z_{w0} - z_w$ where z_w is the vertical distance of the (instantaneous) waterline from the origin (therefore positive ζ corresponds to extra immersion). Also, let the positive pitch angle ϑ be by the stern. A key step is to determine how the two partial derivatives are influenced by hull geometry. Summarizing the investigation of Paulling & Rosenberg [1959], these derivatives are expressed as follows. The derivative with respect to heave is

$$
-2\rho g \int_{x_s}^{x_b} y_{w0}^2(x) \left.\frac{\partial y}{\partial z}\right|_{z=z_{w0}(x)} dx + \rho g\, z_{w0} A_{wo} \tag{78}
$$

and the one with respect to pitch is

$$
2\rho g \int_{x_s}^{x_b} x\, y_{w0}^2(x) \left.\frac{\partial y}{\partial z}\right|_{z=z_{w0}(x)} dx - \rho g\, z_{w0} A_{w0} x_F \tag{79}
$$

In the above, A_{w0} is the still waterline and x_F is the longitudinal coordinate of the centre of flotation. Also, x_s, x_b are longitudinal coordinates of the furthest points at the stern and bow respectively. It is immediately noticed that flare, as represented by the derivative $\partial y/\partial z$ is very important. Furthermore, deck submergence in steep waves as well as a wide transom should enhance the fluctuation of restoring incurred by the heave and pitch motions.

Several semi-coupled or fully coupled models of heave, pitch and roll, for longitudinal waves, have been investigated in the past (see, for example, Blocki [1980], Hua [1992], Oh, Nayfeh & Mook [1992], Neves [2002]). In principle, the mathematical model should account for direct excitations in heave and pitch and for the parametric one in roll. Inspection of stability of this system leads to taking its linear variational vector equation, which is essentially a set of three 2nd-order o.d.e. s with time-dependence in the coefficients of damping and restoring. For these it is known that additional instabilities (and thus "resonances" of the original system) arise for ω_e near to sub-multiples of the sum and difference of the natural frequencies of the participating modes (see, for example, Hsu [1963]; [1965], Neves [2002]), provided that "sufficient" parametric amplitudes are present. In our case, given that the natural frequencies of heave and pitch are usually higher than that of roll, the eligible ω_e should be in the vicinity of $\frac{\omega_0^r + \omega_0^p}{j}, \frac{\omega_0^r + \omega_0^h}{j}, \frac{\omega_0^p - \omega_0^r}{j}, \frac{\omega_0^h - \omega_0^r}{j}$ where $j = 1,2,3,...$ Assuming that the pitch and heave natural frequencies are considerably larger than (e.g. double) the roll natural frequency, taking j = 1 should create resonances near the fundamental one (resulting from the two subtractions) and two lower than the principal (with reference to the frequency axis $a = \omega_0^2/\omega_e^2$). Those near the fundamental should not be relevant for a head-sea scenario. But the others, which correspond to a high frequency of encounter, may not be ruled out. With a similar thinking, it is highly unlikely that the $j > 1$ resonances can be realised. Neves [2004] suggested recently that the linear roll variational equation is, after substitution of the linear heave and pitch responses, a Hill- type equation rather than a Mathieu one as customarily believed.

10. EFFECT OF SURGING IN FOLLOWING SEAS

Very often parametric instability is examined assuming that, despite the wave passage, the forward motion of the ship is unaffected. This assumption could be off the truth when a ship is sailing in steep following waves with a nominal speed that falls in the region of the so-called large amplitude surging motion. This speed region has been identified by IMO to be approximately between $(1.4 - 1.8)\sqrt{L}$ (in knots). The characteristic of this behaviour is the large and asymmetric fluctuation of surge velocity. For an observer moving with the wave celerity, the ship should appear to be overtaken by the wave quickly when its middle is near a trough, but very slowly when it lies near a crest (in absolute terms the speed of the ship will be highest around the crests and lowest near the troughs). As a matter of fact, the assumption that a ship transits from all positions of the incident wave with the same speed should be, in this case, inappropriate. The solution is to adopt a "position-dependent" representation of the fluctuating restoring, instead of the ordinary time-dependent one. But such a substitution of the time-dependent restoring by a position-dependent one means that the equation of surge motion should be considered simultaneously; i.e. a coupled model is pertinent [Spyrou 2000]

(Roll)

$$
\ddot{\varphi} + 2k\dot{\varphi} + \omega_0^2[1 - h\cos(kx)]\varphi - n\omega_0^2\varphi^3 = 0 \tag{80}
$$

(Surge)

$$(m - X_{\dot{u}})\dot{U} = T(x,U;n;Ak,\lambda) - R(x,U;Ak,\lambda) - X_w(x;Ak,\lambda) \quad (79)$$

where $m, -X_{\dot{u}}$ stand respectively for ship mass and surge added mass, the functions T, R represent the propeller thrust and ship resistance, respectively, X_w is surge wave force and Ak is wave slope. Given that $\dot{x} = U - c$ where c is the wave celerity, we essentially have a system of two o.d.e.s with respect to φ, x that can be easily solved in the time domain.

The coupling with surge should alter the layout of the instability regions. Indicatively, in figure 12 the theoretical instability regions are compared with and without surge coupling, taking as a basis a fishing vessel that had been known to show tendency for large amplitude surging. This vessel was not prone to principal parametric resonance but it seemed theoretically possible to display higher parametric resonances in a very extreme environment. The effect incurred upon the resonance regions is quite obvious. Further investigations with other ships should reveal the full potential of this effect.

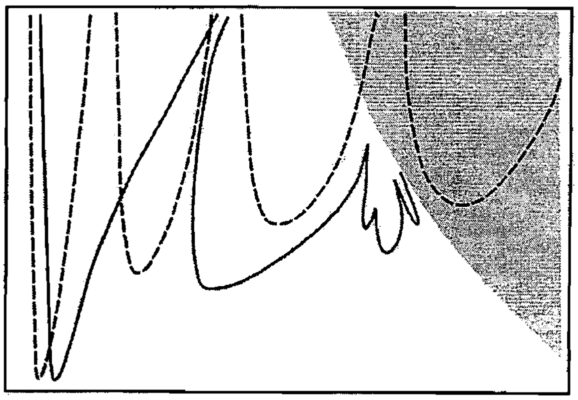

Figure 12. Layout of parametric instability regions with (continuous line) and without (dashed) surge coupling, for a fishing vessel. The grey area corresponds to surf-riding (a possibility only for the coupled model). Note that the principal resonance did not exist for this vessel, i.e. the first shown resonance for each scenario corresponds to the fundamental one. The higher resonances required excessively large h.

11. CONCLUDING REMARKS

According to the review of the field and the new findings presented above, it appears that there is currently sufficient understanding for developing scientifically sound as well as practical design criteria for parametric rolling. ABS's [2004] criteria move into this direction, yet their neglect of the probabilistic character of the seaway, might lead to requirements that are expensive and tough to meet at the design stage. As has been proposed here, a criterion that combines an assessment of transient roll response with the "groupness" characteristic of extreme waves should be more practical compared to a classical criterion deduced from the condition of asymptotic stability of the upright state of the ship, since the latter presupposes a finely tuned critical wave group with infinite run length. Another advantage of the probabilistic viewpoint is that it could be easily integrated within a risk assessment methodology; because the probability of parametric rolling becomes equal to the probability of encounter of a critical wave group. This could be obtained from current theoretical or parametric models of probability distributions of wave parameters. Knowledge of the critical wave group leads directly to the definition of the least roll damping and the tolerable restoring fluctuation.

Coupled motions should receive more attention. At the moment, their quantitative effect on the classical prediction formulae for parametric rolling, which are derived from single degree-of-freedom models, is still uncertain although the available theoretical basis is quite complete. Complex numerical codes could prove useful in this direction, provided that a "black-box" approach is not followed; i.e. the scope, limitations and assumptions of these codes concerning parametric rolling phenomena should be comprehended before deriving conclusions.

Lately, a discussion has been initiated about the effectiveness of standard seakeeping methods in ascertaining, from model experiments or numerical simulations, that a ship is safe from parametric rolling [Belenky *et al.* 2003]. In particular, the assumption of ergodicity of parametric rolling has been questioned and subsequently, the validity of deriving conclusions from single temporal averages of roll response (especially as these are usually of limited time, e.g 30 mins at full scale). It appears that a different approach to this problem that could cope with both the nonlinearity that is responsible for the observed finite roll oscillations and the fact that the wave "groupness" (i.e. time localized non-stationary features of the wave field) is the essential instigator of the phenomenon.

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