THE ROLE OF MATHIEU'S EQUATION IN THE HORIZONTAL AND TRANSVERSE MOTIONS OF SHIPS IN WAVES: INSPIRING ANALOGIES AND NEW PERSPECTIVES

Dr. K.J. Spyrou

Centre for Nonlinear Dynamics and its Applications
University College London
Gower Street, London, WC1E 6BT
email: k.spyrou@ucl.ac.uk

ABSTRACT

The paper discusses the occurrence of parametric-type instabilities for different ship motions in waves. It is shown that, although Mathieu instability is usually associated with roll motion, Mathieu's equation is present in the yaw dynamics as well. More importantly, this equation seems to underlie broaching instability at relatively low speed. Froude number ranges where this is likely to arise are then derived and a number of interesting analogies with roll motion are pointed out. This connection serves as an introduction to the second part of the paper which is devoted to the problem of parametric resonance of roll. A number of promising directions of research in this area are discussed.

INTRODUCTION

At least since the early fifties, Mathieu's equation has been recognized to play an important role in ship dynamics (Weinblum & StDenis, 1950; Grim, 1952). In this paper we examine in parallel the yaw and roll motions in following waves and we point out a number of very interesting similarities that exist in the analysis of these motions. For yaw, we show that the existence of the autopilot can give rise to parametric type instability. Expressions for the natural frequency and damping of the steered ship with proportional-differential control are derived. Furthermore, simple relationships linking Froude number and wave characteristics on the instability boundaries are presented. This information can be directly utilized in early ship design.

In respect to the roll problem, we review initially the general approaches and we present a unifying framework for parametric resonance and pure-loss of stability. Specific technical aspects that are discussed in the paper are the following: (a) Improved linear analysis for pure-loss, (b) analysis of some recent experimental data for parametric resonance (c) nonlinearity of restoring (d) combined parametric and direct excitation, (e) effect of surge nonlinearity, (f) effect of rudder and yaw-roll coupling.

FUNDAMENTAL ASPECTS OF YAW DYNAMICS IN LARGE WAVES

Consider the simplified linear yaw response model of Nomoto and let's introduce at the right-hand-side angle dependent sinusoidal excitation in order to account, in a qualitative sense, for the effect of the waves on yaw motion

$$T'\dot{\psi}'' + \dot{\psi}' = K'\delta + A'\psi \cos(\omega'_e t' - a) \tag{1}$$

As usual, K, T are system gain and time constants, ψ is heading angle in respect to the wave direction and is assumed relatively small, δ is rudder angle, A is wave excitation amplitude, ω_e is the encounter frequency, t is time and a is a phase angle. The prime indicates nondimesional quantitities (t is nondimensionalized in respect to U/L where U is the forward speed and L is the length of the ship). Consider further rudder control with a linear law based on proportional and differential gains k_1 and k_2:

$$\delta = -k_1(\psi - \psi_r) - k_2' r' \tag{2}$$

Above, ψ_r is the prescribed heading and r is the rate of turn. By introducing (2) into (1) and dropping for simplicity the phase angle a, we obtain after rearrangement:

$$\ddot{\psi}' + \frac{(1+k_2'K')}{T'}\dot{\psi}' + \frac{k_1K'}{T'}[1 - \frac{A'}{k_1K'}\cos(\omega_e' t')]\psi = \frac{k_1K'}{T'}\psi_r \tag{3}$$

or,

$$\ddot{\psi}' + \gamma\dot{\psi}' + {\omega'_{0(yaw)}}^2[1 - h\cos(\omega_e' t'))]\psi = j \tag{4}$$

where:

$\omega'_{0(yaw)} = \sqrt{\frac{k_1K'}{T'}}$, $\gamma = \frac{1+k_2'K'}{T'}$ (damping), $h = \frac{A'}{k_1K'}$ (amplitude of parametric variation of restoring), $j = \frac{k_1K'}{T'}\psi_r$.

It is easily recognized that (4) is Mathieu's equation with the addition however of bias-like external static forcing, j. By introducing the time variable transformation $\tau = \omega_e' t'$ we obtain:

$$\frac{d^2\psi'}{d\tau^2} + \frac{\gamma}{\omega_e'}\frac{d\psi'}{dt} + \frac{{\omega'_{0(yaw)}}^2}{\omega_e'^2}[1 - h\cos\tau]\psi = \frac{j}{\omega_e'^2} \tag{5}$$

By rewriting (3) on the basis of heading error $\psi_1 = \psi - \psi_r$, we obtain the alternative form:

$$\ddot{\psi}_1' + \frac{(1+k_2K')}{T'}\dot{\psi}_1' + \frac{k_1K'}{T'}[1 - \frac{A'}{k_1K}\cos(\omega_e' t')]\psi_1 = \frac{A'}{T'}\psi_r\cos(\omega_e' t') \tag{6}$$

or, with $d = \frac{A'\psi_r}{T'}$,

$$\ddot{\psi}_1' + \gamma\dot{\psi}_1' + {\omega'_{0(yaw)}}^2[1 - h\cos(\omega_e' t')]\psi_1 = d\cos(\omega_e' t') \tag{7}$$

As noticed, in (7) there is parametric as well as independent periodic forcing. If (7) is further converted into the standard form of the equation of the mechanical oscillator, with the aid of the transformation $s = \omega'_{0(yaw)}t'$, we obtain:

$$\frac{d^2\psi_1'}{ds^2} + b\frac{d\psi_1'}{ds} + [1 - h\cos(\Omega s)]\psi = f\cos\Omega s \tag{8}$$

where $\Omega = \frac{\omega_e'}{\omega'_{0(yaw)}}$ and $f = \frac{d}{{\omega'_{0(yaw)}}^2}$. In (8) the damping b is basically two times the so-called damping ratio ζ and is given by the expression $b = 2\zeta = \frac{1+k_2K'}{\sqrt{\frac{k_1K'}{T'}}}$. Notably, in this last expression k_1 is also present, since the linear restoring is assimilated into the damping term.

The damping of the unsteered system will be $\frac{1}{T'}$. The dynamic stability of the unsteered vessel in stillwater is governed by the damping $\frac{1}{T'}$ of the open-loop system, basically the sign of T', where $T' > 0$ means stability. However, large positive T' implies slow convergence towards the corresponding steady rate-of-turn the location of which (always for small angles or rates) is generally 'dictated' by the value of the static gain K'. A trend exists for large T' to appear in conjunction with large K' which gives a nearly straight-line *spiral curve*. The effect of active control on damping is represented by the quantity $k_2' \frac{K'}{T'}$ as a result of the presence of a differential gain term in the autopilot. If $T' < 0$, suitable choice of k_2' can turn the damping of the system positive since k_2' multiplies the positive quantity $\frac{K'}{T'}$, thereby yielding stability for the steered ship in calm sea. The wave effects are lumped into the restoring and independent-periodic-forcing terms of (7) since the quantities K' and T' were assumed, at first instance, unaffected by the wave.

Consider now a gradual increase of the propeller revolutions for a small ship sailing in a sea of large following waves. It is well known that surge motion will develop from almost perfectly sinusoidal into a motion where the ship remains for relatively longer time in the region of the wave crest while passing quickly from the trough. This is the so-called *large-amplitude-periodic-surging* phase, that represents basically the forerunner of the stationary condition of *surf-riding*. The effects incurred on the lateral motions by the nonlinearity of surge cannot be accounted for by the single-degree model because there the surge velocity is assumed constant. Nevertheless it is possible to imagine how instability of yaw will come about: The most obvious scenario corresponds to the static case, whereby the amplitude of parametric forcing h becomes so large that it exceeds 1.0. Then the restoring in yaw of the system will turn negative in a certain region of the wave trough since there the wave yaw moment plays a destabilizing role. If this condition is realized during the large-amplitude-surging phase, the ship may sometimes not escape, due to the fact that it passes very quickly from the dangerous, in a yaw sense, trough region. However, if the ship is captured in surf-riding, there will be enough time for realizing such an escape. An analogy may be drawn here between the above statical instability of horizontal motions and the well known *pure-loss of stability* of roll that will be further discussed in the next Section. However in the roll case, negative restoring can appear around the crest of the wave.

Theoretically, instability may also arise however for $h \ll 1.0$, even for h near zero, and the thought-provoking analogy between horizontal motions and roll can be taken further through the so-called *parametric* - route. The second-order differential equation of heading was shown to be basically a damped Mathieu-type equation which is known to exhibit unstable behaviour in certain ranges of parameter values. We shall consider (4) further for the simplest case where the prescribed course is the purely following sea, i.e. $\psi_r = 0$. The net effect is that the independent periodic forcing is eliminated and the results of the classical analysis of Mathieu's equation can find direct application.

In the absence of direct external forcing, we can identify the stability transition curves of (7) by considering a solution in the form $\psi_1' = \Psi e^{\mu t'} \cos(\omega_e' t' + \psi_0')$. This is then substituted back into the original equation and harmonic balance is carried out. Dropping the higher frequency terms, the multiplicative coefficients of $e^{\mu t'} \cos(\omega_e' t')$ and $e^{\mu t'} \sin(\omega_e' t')$ must be zero if the equation are to be valid for all t'. This gives a pair of homogeneous linear equations and the condition for stability can be derived by asking the determinant of this system to be zero. The stability chart of Mathieu's equation is well known. On the so called Strutt diagram, this equation is associated with 'tongue' like instability domains. For the undamped Mathieu's equation the instability regions present their vertices on the $h = 0$ axis, there where $\frac{\omega_e'^2}{\omega_{0(yaw)}'^2} = \frac{4}{n^2}$; n is any positive integer. However when damping is present, a non-zero h is needed in order to destabilize the parametric oscillator. The mimimal destabilizing h is very sensitive to the amount of damping. It can be shown on the basis of the earlier described procedure that the lowest order approximation of the boundary curve in the principal resonance domain is given by: $\frac{h^2}{4}=(1-\frac{\omega_e'^2}{4\omega_{0(yaw)}'^2})^2+\gamma^2\frac{\omega_e'^2}{4\omega_{0(yaw)}'^4}$. At exact resonance, the minimal h required in the first instability region ("*principal* ", $\frac{\omega_e'}{\omega_0'}=2$) will be: $h=\frac{2\gamma}{\omega_{0(yaw)}'}$. However one should note that the minimal h that is required for instability in this region will shift slightly off resonance at: $\frac{\omega_e'}{\omega_{0(yaw)}'}=2\sqrt{1-\frac{\gamma^2}{2\omega_{0(yaw)}'^2}}$. Therefore, only if γ is a small quantity the expression $h=\frac{2\gamma}{\omega_{0(yaw)}'}$ can be assumed to give the minimal destabilizing forcing. Otherwise one should use the expression $h=\frac{2\gamma}{\omega_{0(yaw)}'}\sqrt{1-\frac{\gamma^2}{4\omega_{0(yaw)}'^2}}$. Relationships for the *fundamental* (n=2) and the third (n=3) resonance can be found, for example, in Nayfeh and Mook (1979).

The interpretation of the principal resonance condition of the damped system, in terms of the K',T' indices is as following: The inequality $h<\frac{2\gamma}{\omega_{0(yaw)}'}$ is equivalent with : $\frac{A'}{k_1K'}<2\frac{\frac{1+k_2'K'}{T'}}{\sqrt{k_1\frac{K'}{T'}}}$. The minimal gains for a dynamically stable ship in calm sea ($K',T'>0$) are given by: $\sqrt{k_1}(1+k_2'K')=\frac{A'}{2\sqrt{\frac{K'}{T'}}}$.

This indicates that the differential gain should have a special influence on parametric instability. We can contrast this with the condition of statical instability that arises in surf-riding: Assuming $T'>0$, it would suffice to have positive restoring everywhere, that is, $1-h>0$. Since $h=\frac{A'}{k_1K'}$ the critical gain value k_1 is: $k_1=\frac{A'}{K'}$.

Dependence of the instability domain upon the Froude number

For overtaking waves the frequency of encounter is positive and the condition of exact resonance is written as: $\frac{\omega_e'}{\omega_{0(yaw)}'}=\frac{2}{n}$, n=1, 2, 3,... . Thus with increasing n the vertices will accumulate near to zero frequency of encounter. For ψ_r = 0 we can write $\omega_e'=\frac{2\pi L}{\lambda}(\frac{c}{U}-1)$. Then, with the substitutions $\omega_e'=\frac{2\omega_{0(yaw)}'}{n}$ and $\frac{c}{U}=\frac{Fn_{wave}}{Fn}$ (Fn_{wave} is the Froude number corresponding to wave celerity) we obtain the parametric equation of the vertices of the undamped system: $Fn = \frac{Fn_{wave}}{1 + \frac{\lambda \omega'_{0(yaw)}}{L\pi n}}$. Given that $Fn_{wave} = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda}{L}}$ we arrive finally at the expression: $Fn = \frac{\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda}{L}}}{1 + \frac{\omega'_{0(yaw)}(\frac{\lambda}{L})}{\pi n}}$.

Let's consider now the domain of variation of the yaw natural frequency $\omega'_{0(yaw)} = \sqrt{\frac{k_1 K'}{T'}}$. On the basis of the extensive data collection of Barr et al (1981) the ratio $\frac{K'}{T'}$, which is often regarded as a measure of the initial turning ability of a ship (one should remind here that the turning-index P of Norrbin is approximately equal with $\frac{1}{2}\frac{K'}{T'}$), may be taken to vary in the range [0.3 - 1.4]. Consequently $\omega'_{0(yaw)}$ should lie in the range $[0.55\sqrt{k_1}$ - $1.18\sqrt{k_1}]$. If a proportional gain k_1 between 1.0 and 2.0 was selected then $\omega'_{0(yaw)}$ should be between 0.55 and 1.67. In Fig. 1 we have plotted the critical Fn versus $\frac{\lambda}{L}$ for $\omega'_{0(yaw)}$ = 0.5, 1.0 and 1.5. It must be pointed out however that, yaw motion is highly damped and, in reality, large parametric variation h will be required in order to reach the instability domains. This problem is currently under investigation.

Confirmation with a multi-degree model

In order to confirm that such instability is exhibited also by more detailed models of ship manoeuvring in waves, we have carried out extensive numerical studies based on a nonlinear surge-sway-yaw-roll model, see Spyrou (1996). The results were reported in Spyrou (1997). A key finding is the emergence of subharmonic response at a critical combination of prescribed heading angle, wave steepness and length/ratio. This leads further to a dangerous jump to resonance from a fold point of the amplitude response curve.

A UNIFYING FRAMEWORK FOR STUDYING THE EFFECT *OF GZ*-VARIATION IN A LONGITUDINAL SEAWAY

Overview and possible new directions

Two principal viewpoints exist for studying the dynamics of ship rolling in a longitudinal seaway: According to the first, the seaway is represented in the roll equation through a time-varying roll righting arm and the ship is assumed sailing in a condition of quasi-static equilibrium in terms of heave and pitch (Grim 1952; Kerwin 1955; Paulling & Rosenberg 1959; Paulling 1961). Instability of static type is then possible when the righting arm reduction, usually around the crest, is so severe that, in certain regions there is complete absence of restoring capability. If, as sometimes happens in a following sea, the frequency of encounter ω_e between the ship and the wave is "sufficiently" low, a slight disturbance in roll applied when the ship's centre of gravity approaches a crest can grow exponentially until capsize, the so called *pure-loss of stability*. The presence of time-periodic restoring can also create instability of parametric type: The undamped rolling motion around the upright equilibrium can be described at first instance by Hill's equation $\ddot{\phi} + P(t)\phi = 0$ where ϕ is the roll angle; $P(t)$ is a time-periodic function whose Fourier expansion is $P(t) = \sum_i P_i \cos(i\omega_e t + a_i)$ where P_i and a_i are respectively amplitude and phase of the ith consituent harmonic. For simplicity it is quite common to assume that $P(t)$ is a circular function, i.e. retain only the first term in the Fourier series. Then, $P(t) = P_0 + P_1\cos(\omega_e t)$ and Mathieu's equation, $\ddot{\phi} + \omega_{0(roll)}{}^2[1 + h\cos(\omega_e t)]\phi = 0$ is obtained, where h is again the amplitude of parametric variation, this time however in roll restoring; and $\omega_{0(roll)}$ is the natural frequency in roll. Of course, the average restoring between crest and trough usually differs from the still-water restoring and better approximations are sometimes necessary (see Hamamoto & Panjaitan, 1996).

The second viewpoint allows the roll and pitch modes to interact. The scope of this approach is obviously wider than the one discussed earlier. Here the pitch motion is not prescribed and phenomena of dynamic coupling between pitch and roll are not suppressed. The growing roll motion can create substantial pitch that, in turn, will affect back roll (Nayfeh, 1988). In this approach no limitation exists about the frequency of encounter. However, the several uncertainties that exist about the form of the mathematical model grant these studies with a character inherently qualitative. Nayfeh, Mook & Marshall (1973) have discussed the existence of an *internal* resonance phenomenon when $\frac{\omega_{0(pitch)}}{\omega_{0(roll)}} = 2$ and $\omega_e = \omega_{0(roll)}$ ($\omega_{0(pitch)}$ is the natural frequency in pitch). This is caused by a kind of saturation in pitch response where all the energy that enters the pitch mode is shed into roll. As a result of this, roll becomes resonant. This may be an explanation for the observation of Froude about the development of undesirable roll when $\frac{\omega_{0(pitch)}}{\omega_{0(roll)}} = 2$. Further investigations along these lines have been pursued by several researchers: Sethna & Bajaj (1978) found amplitude modulated motions due to Hopf bifurcations in the averaged equations of coupled roll and pitch. Nayfeh (1988) introduced nonlinear couplings between the two modes and has shown that Hopf bifurcations can arise at the above frequency ratios. Other studies about the coupled motions were by Nayfeh & Oh (1990), and Hua (1992).

In spite of the reference to roll instability, for many years capsize was not directly addressed since the nonlinearity of restoring at large angles of heel was not taken into account. When the trivial state $\phi = 0$ becomes unstable it is not certain that capsize will occur; because the orbit diverging from the vicinity of the upright equilibrium can reach an oscillatory steady-state, the existence of which is justified from the nonlinearity of the righting moment, remaining therefore bounded. A more physically appealing model, featuring nonlinear restoring and also nonlinear damping, was used by Blocki (1978). Blocki has also interfaced the traditional analysis with the random character of real waves by considering the probability of encountering a wave train that has the potential to create parametric instability. Feat & Jones (1981) investigated the effect of steady heel.

From numerous studies on parametric instability is known that the transition lines represent the locii of supercritical (left branch) and subcritical (right) bifurcations. The supercritical bifurcation creates stable subharmonic oscillations whereas the subcritical give rise to unstable ones. Thus at the subcritical there will be a discontinuous jump, often destined for the subharmonic oscillations originating from the supercritical. Nayfeh and his coworkers (Nayfeh & Sanchez 1988, Zavodney, Sanchez & Nayfeh 1989 & 1990, Zavodney and Nayfeh 1989), have used the analytical method of multiple scales in order to approximately locate stability boundaries due to saddle-nodes and period-doubling bifurcations of these periodic orbits in respect to the principal and the fundamental resonance of the parametric oscillator. They derived also these boundaries numerically. Soliman & Thompson (1992) presented a complete bifurcation diagram around the principal resonance for an equation with quadratic, softening-type nonlinearity in restoring. Sometimes the oscillations penetrate into the domain of stability of the trivial solution giving rise to bistability. The posibility of safe, unsafe and also indeterminate due to a tangled subcritical bifurcation, jumps has also been shown. In another study, Kan and Taguchi (1992) carried out extensive studies based on integration from a grid of initial conditions in phase-space (ϕ, $\dot{\phi}$), including also direct wave excitation. They distinguished between the highly unlikely event where the vanishing angle remains constant in spite of the changes in restoring from trough to crest, and the more realistic scenario where the vanishing angle is allowed to varied as well.

Although pure-loss and parametric resonance are usually treated as different types of instability, a closer examination of Mathieu's equation should reveal that such a distinction is in fact artificial. At first instance, in the Strutt diagram the domain of pure-loss should occupy the region above $h = 1$ with $\frac{\omega_e}{\omega_{0(roll)}} \rightarrow 0$.

However, distinguishing between the two in a more rational way is not a trivial task because what constitutes a pure-loss case may be perceived in a number of different ways. For example, one may characterize as pure loss the monotonic increase of the angle of heel until the ship is overturned ; or, that capsize where the ship has shown no resistance to overturning (negative restoring throughout).

Another overlooked effect in the development of parametric roll is this due to the nonlinearity of surge. Here the interesting point is that although there is no coupling between roll and surge, the effect comes through the time variable. The net effect of surge nonlinearity is a virtual rescaling of the time axis for the other degrees of freedom. It is implied that ω_e should rather be written as a time periodic function $\omega_e(t)$, the form of which for a given ship can vary depending on propeller thrust and the prevailing wave characteristics. If the frequencies of encounter of pure-loss or parametric resonance overlap with the frequencies where the nonlinearity of surge is significant, this nonlinearity can be the key factor for capsize because it undermines safety in roll since the ship spends longer time around the dangerous crest region.

We should note finally the existence of literature on stochastic parametric excitation either from a ship's point of view (Dunwoody, 1989) or from the wider angle of general dynamics (Ibrahim, (1984). Such a viewpoint is however outside the scope of the present paper.

A modified linear theory for "pure-loss" type capsize

It is rather well known that the condition of instantaneous negative restoring does not suffice for realizing pure-loss capsize. If the ship moves "too quickly" into a wave region where restoring becomes increasingly positive, then the initial tendency to heel will be counteracted by a new tendency for returning to the upright. Albeit the ship may still capsize in a typical parametric instability fashion, this will not represent realization of the static in nature phenomenon of pure-loss. A true pure loss would further entail that:

(a) The ship is given enough time to "adjust itself" in order to achieve the condition of quasi-static equilibrium in heave and pitch; i.e. the natural periods of these two modes, whose values are often comparable, should be considerably lower than the encounter period.

(b) The time spent by the ship near the crest should be sufficiently long in order to allow development of heel up to a very large angle without experiencing serious resistance and tendency for returning to the upright.

Paulling (1959) suggested that, assuming that the natural period of roll is at least two times the natural period of the other two modes, both conditions can be marginally satisfied if the time spent around the crest is at least equal with half the natural period of roll. However the linear natural roll period depends on the metacentric height (GM) and obviously has no physical meaning when (GM) is negative. To improve the analysis in respect to consideration (b), let's consider at first instance the linear equation of unforced roll motion:

$$(I_X + J_X)\ddot{\phi} + B_0\dot{\phi} + W(GM)\phi = 0 \qquad (9)$$

where I_X, J_X are, roll moment of inertia and added moment of inertia rerspectively, B_0 is damping, W is the weight of the ship and ϕ is the roll angle. It is more convenient for our analysis to write this equation as:

$$\ddot{\theta} + \beta\dot{\theta} + c\theta = 0 \qquad (10)$$

where $\beta = \dfrac{B_0}{(I_X + J_X)}$, $c = \dfrac{W(GM)}{(I_X + J_X)}$ (if $c > 0$ then $c = \omega_{(roll)}{}^2$) , $\theta = \dfrac{\phi}{\phi_v}$ where ϕ_V is the angle of vanishing stability in still water. The solution of (2) is given by : $\theta = a_1 e^{\frac{-\beta-\sqrt{\beta^2-4c}}{2}t} + a_2 e^{\frac{-\beta+\sqrt{\beta^2-4c}}{2}t}$

If the stiffness coefficient $c < 0$ then the second term of the solution is unbounded, in other words the trivial solution $\theta = 0$ is unstable. To define a_1 and a_2 we need to specify the initial conditions of the ship at $t = t_0$.

Obviously it is convenient to select $t_0 = 0$. To allow the motion to diverge from the unstable solution we also assume that the ship has an initial small roll velocity, say ε (for example ε=0.01), so $\frac{d\theta(0)}{dt} = \varepsilon$ while θ (0)=0. Then the coefficients of the solution are given by : $a_1 = \frac{-\varepsilon}{\sqrt{\beta^2 - 4c}}$, $a_2 = \frac{\varepsilon}{\sqrt{\beta^2 - 4c}}$

Concentrating on the unbounded term of the solution we can derive the required time, t_{pl} , to capsize by setting θ =1 and solving for t : $t_{pl} \geq t = \frac{-2\ln\frac{\varepsilon}{\sqrt{\beta^2-4c}}}{-\beta + \sqrt{\beta^2 - 4c}}$. Assuming that this time should correspond, at maximum, to half encounter wave period (from the node of the wave's up-slope to the node of the down-slope) then the highest required frequency of encounter is $\omega_e = \frac{2\pi}{2t_{pl}}$ and in nondimensional form $\omega_e' = \omega_e\sqrt{\frac{L}{g}} = \frac{2\pi}{2t_{pl}}\sqrt{\frac{L}{g}}$, where L is ship length and g is the acceleration of gravity. We note that here the nondimensionalization is carried out in a different way compared to the analysis of yaw presented in Section 2. The minimal Fn that "allows" capsize can be recovered from the expression $\omega_e' = \frac{2\pi L}{\lambda}(Fn_{wave} - Fn)$.

Some practical aspects of the parametric instability of roll

The occurrence of capsize due to parametric resonance is physically recognizable by the ever growing oscillatory roll that precedes capsize. The true realization of a capsize according to this mechanism requires however a favourable combination of ship and wave parameters the existence of which is only in a few times possible. As a worked example, let's try to identify the relation between nominal Froude number and wave length for the vertices of the transition lines of the linear and undamped Mathieu equation. On these vertices, the relation $\frac{\omega_e^2}{\omega_{0(roll)}^2} = \frac{4}{n^2}$, n=1,2,... should be satisfied. For an overtaking following sea this reduces to $\frac{\omega_e}{\omega_{0(roll)}} = \frac{2}{n}$ which, in terms of Froude number, yields further : $Fn = Fn_{wave} - \frac{\omega'_{0(roll)}\lambda}{\pi n L}$ or, $Fn = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda}{L}} - \frac{\omega'_{0(roll)}\lambda}{\pi n L}$. In Fig. 2 is shown the relation between Fn and (λ/L). To prove the usefulness of this analysis, two characteristic examples of ships that have been tested earlier experimentally in Japan will be considered : A container (L=150.0 m , $\omega_{0(roll)} = 0.144, \omega'_{0(roll)}$=0.566) and a purse-seiner (L=34.5 m, $\omega_{0(roll)} = 0.840$, $\omega'_{0(roll)}$=1.577), see Umeda et al. (1995), Hamamoto et al. (1995). It had been observed during the free running tests that, in quartering seas the containership exhibited parametric instability (for example at Fn=0.23). This never happened though for the purse-seiner. In the Table below, we have calculated the Froude numbers which correspond to the vertices of the first four resonances. It is clear that, for the purse-seiner parametric instability could only arise in connection with the second, third, fourth etc. regions. The main reason for not having observed such resonant motions during the experiments is primarily the existence of roll damping, since energy dissipation has generally a stabilising effect on one-degree-of freedom systems.

	λ/L=1.0		λ/L=2.0	
	container	purse-seiner	container	purse-seiner
n=1	0.219	<0	0.203	<0

n=2	0.309	0.148	n=2	0.387	0.062
n=3	0.339	0.232	n=3	0.444	0.229
n=4	0.359	0.274	n=4	0.479	0.313

Let's consider now the linear roll equation in the presence of parametric excitation and $2\tau = \omega_e t$:

$$\frac{d^2\theta}{d\tau^2} + \frac{2\beta}{\omega_e}\frac{d\theta}{d\tau} + 4\frac{\omega_{0(roll)}^{\ 2}}{\omega_e^{\ 2}}[1 + h\cos(2\tau)]\theta = 0 \qquad (11)$$

By applying the variable transformation $\theta = we^{\frac{-\beta}{\omega_e}\tau}$ (McLachlan, 1946) we can obtain an equivalent equation where damping is eliminated as such, and is transferred into the restoring term:

$$\frac{d^2w}{d\tau^2} + [4\frac{\omega_{0(roll)}^{\ 2}}{\omega_e^{\ 2}} - \frac{\beta^2}{\omega_e^{\ 2}} + 4\frac{\omega_{0(roll)}^{\ 2}}{\omega_e^{\ 2}}h\cos(2\tau)]w = 0 \qquad (12)$$

Damping modifies slightly the natural frequency (from $\omega_{0(roll)}$ to $\sqrt{\omega_{0(roll)}^{\ 2} - \frac{\beta^2}{4}}$) and reduces the growth rate of θ by β. Primarily however damping tends to make the transition lines curved and increases very substantially the minimum parametric excitation h required for instability, particularly in respect to the higher-order resonances. This effect of damping on h_{min} seems to have neen overlooked by Kerwin (1955) since he concluded that the damping does not affect the resonance curves.

The linear damping coefficient can be easily derived if the extinction curves of the ship are known. The relationship is $\frac{\beta}{\omega_{(roll)}} = -\frac{2}{\pi}\ln(1-a)$ or $\beta = \frac{-4}{T_0}\ln(1-a)$, where a represents the average decrement of the maximum roll angle within one cycle during a roll-decay test (see for example Kerwin, 1955). On the basis of the values of -ln (1-α) ("effective extinction coefficient") is derived that $\beta = 0.117$ (purse-seiner, corresponding to a damping ratio ζ=0.069) and $\beta = 0.025$ (container, ζ=0.086), Hamamoto et al. (1995). It is then possible to work out, for λ/L=1.0, the minimal h that can cause instability in the first and second region for the container and in the second region for the purse-seiner by using for example the expressions in Nayfeh & Mook (1979). The fundamental resonances of the two ships require h near to 1.0 or higher. Such values correpond of course to high waves where the exact type of capsize may be hardly distinguishable, particularly if additional effects, such as water shipped on deck are present. Given that the analytical approximation of the transition curves will not be very accurate as h becomes large and the damping is not very near to zero, a more practical option is to find the stable/unstable domains with a numerical procedure, Figs. 3 and 4.

Nonlinearity of restoring

Depending on the true nature of the restoring curve of the ship (for example if it is initially softening or hardening) various types of finite amplitude oscillations are possible even without direct excitation in roll.
A familiar nonlinear form of Mathieu's equation is the one with a softening cubic nonlinearity in restoring $R(\theta, t)$:

$$R(\theta, t) = \omega_{0(roll)}^{\ 2}[1 + h\cos(\omega_e t)]\theta - q\theta^3 \qquad (13)$$

A number of researchers have presented analytical (for example McLaclhan, 1956; Skalak & Yarymovych, 1960; Minorsky, 1962) or numerical (Byant & Miles, 1990; Bishop & Clifford, 1994) solutions of nonlinear versions of Mathieu's equation. Useful information about the solutions of (13) can be derived also from the earlier referenced studies of Nayfeh.

Combined parametric and direct excitation in quartering seas

With the sea on the quarter extra effects are incurred upon the ship : Firstly, as the angle ψ between the ship and the direction of wave propagation increases, the encounter frequency ω_e will increase too since $\omega_e=(2\pi/\lambda)\,(c\text{-}U\cos\psi)$. However the amplitude of parametric excitation will tend to reduce. Moreover, the quantity $\sin\psi$ which appears in direct wave loading calculations (in the sense of Froude-Krylov and diffraction excitations) will no longer be zero. As a result, a direct wave roll moment will arise, characterized by its amplitude F and phase θ_1. Due to the simultaneous presence of two types of wave excitation, both the principal and fundamental resonance will be significant. This has been nicely illustrated in terms of capsize boundaries by Kan & Taguchi (1992). These authors fixed the external exitation at a high level and gradually stepped up from zero the parametric amplitude. Of course, if the encounter angle is varied gradually from beam to following sea, the increase in the amplitude of the parametric wave excitation will be accompanied by significant decrement in the amplitude of the direct one. An improved roll equation that can take this fact into account is :

$$\ddot{\theta}+\beta\dot{\theta}+\omega_{0(roll)}{}^{2}\ [1+h\cos\psi\,\cos(\omega_e t)]\theta-q\theta^{3}\}=(F_0/\phi_v)\sin\psi\,\cos(\omega_e t-\theta_1)\qquad(14)$$

In reality the amplitude of direct excitation will be generally less than the $F_0\sin\psi$ where F_0 is the beam sea excitation amplitude, Weinblum & StDenis (1950). However this difference can be considered as a safety margin. It should be noted also that the earlier applied transformation to an equivalent undamped system is no longer feasible due to the existence of the cubic term and the non-zero forcing at the right-hand-side.

The effect of surge

The significance of surge nonlinearity is profound in respect to the phenomenon of pure-loss, Spyrou (1997). Broadly speaking, pure-loss requires that $h\geq 1$, which is normally realized when $\frac{\lambda}{L}>1$ and also ω_e is near zero in order to have enough time for capsize. However, at $\frac{\lambda}{L}=2$, to achieve $\omega_e=0$ the required Fn is 0.564. For many commercial vessels such Froude numbers are rather above their operational range. Even a Froude number near Fn=0.399 (which corresponds to $\frac{\lambda}{L}$ exactly equal to 1.0 and can be taken as a lower limit for $\omega_e=0$) is still too high. On the other hand, if nonlinear surging occurs, the ship should remain for sufficient time around the crest of the wave even though its nominal Froude number might lie at a considerable distance from zero. It is known that given a sinusoidal wave of specific length and height, the large-amplitude-surging type of response appears at a nominal ship speed that is well below the wave celerity c, and subsequently in a region where ω_e is away from zero. Large amplitude surging is likely to lead into surf-riding. However it should be remarked that surf-riding takes place in the region of the trough and as such, it does not pose a direct capsize threat by pure loss. So the very condition that linear theory nominates as the single most dangerous for pure-loss seems to be 'immune' of this capsize mode! The real threat that is associated with the transition to surf-riding is in fact broaching.

The linear approach may be reasonably valid however up to a wave steepness where surf-riding cannot arise. But then it is unlikely that in such, not particularly steep, waves the restoring capability of the ship will be reduced so dramatically that it can generate pure-loss. Quite often the minimum steepness that can give rise to negative restoring at the upright condition lies within the range of steepness where surf-riding can exist. In summary, it seems to be unwise to neglect the effect of this nonlinearity. Since the surge nonlinearity is 'felt' through the increasing importance of higer order harmonics in response, the equation that needs to be studied is Hill's-like with the following specific form:

$$\frac{d^2\theta}{dt^2} + \beta\frac{d\theta}{dt} + \omega_0^2\{[1+\sum_i h_i(\omega_e)\cos(i\omega_e t)]\theta - q\theta^3\} = 0 \qquad (15)$$

In Fig.5 we show the effect that the nonlinearity of surge can have on the shape of the instability boundaries.

The effect of rudder and yaw-roll coupling

The importance of this effect for parametric instability and capsize is unknown at this moment. Several ship types are known to exhibit however coupling of this nature, including containers, ro-ro ferries and fishing vessels. The basic mechanism is that yaw induces roll that, in turn, causes more yaw. Also, an alternating roll moment is induced on the hull directly by the rudder as it oscillates to maintain on average the desired course of the ship. This moment is "felt" in roll if it represents a fair percentage of the righting moment of the ship. In quartering seas ships can perform considerable yawing motions that depend also on the method of steering, Spyrou, 1997. Due to this yawing the encounter frequency rather than being constant, is in fact a periodic function of time. It becomes obvious that deeper study of this mechanism entails the use of a multi-degree mathematical model.

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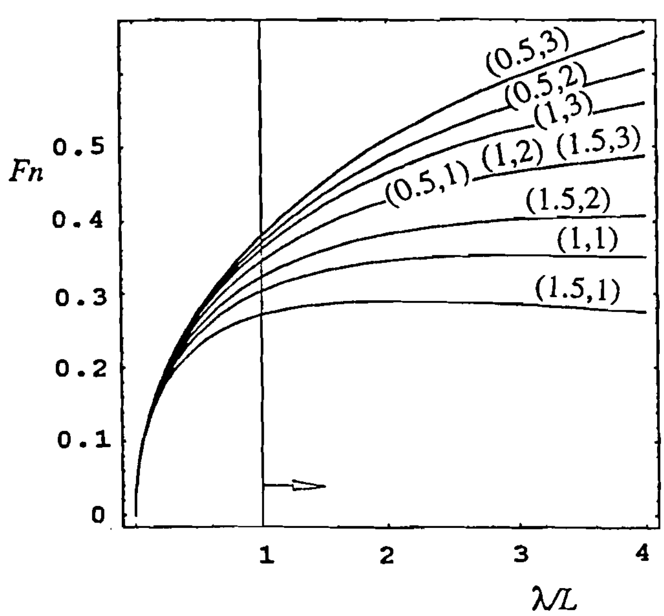


Fig. 1: The condition of exact resonance in yaw arises for certain combinations of λ/L and Fn. Each curve is defined by the pair (ω_0', n).

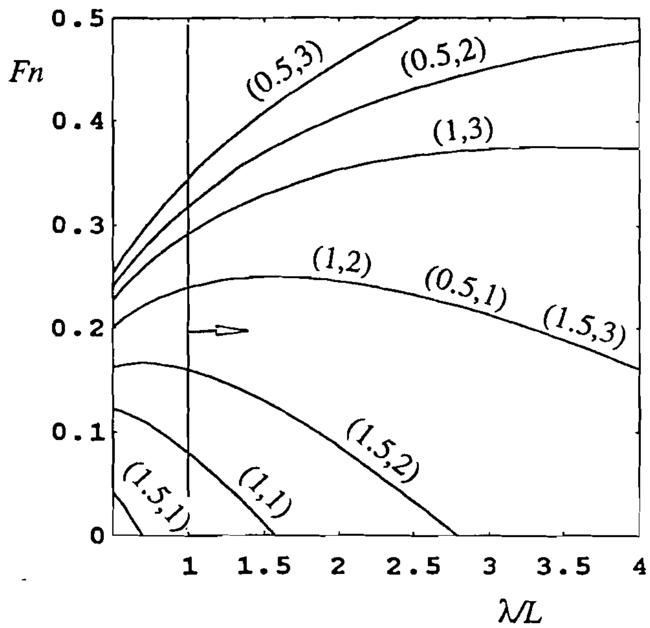


Fig: 2: The loci of exact resonance for roll.

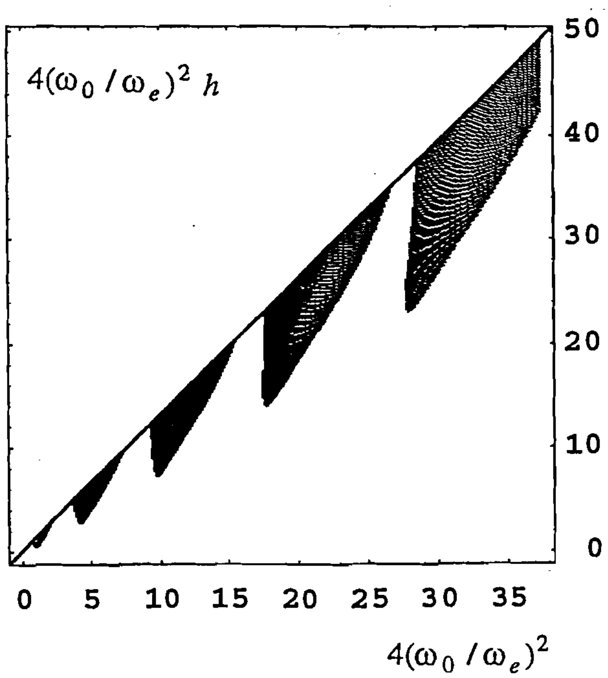


Fig. 3: Numerically derived Strutt diagram, ζ=0.069.

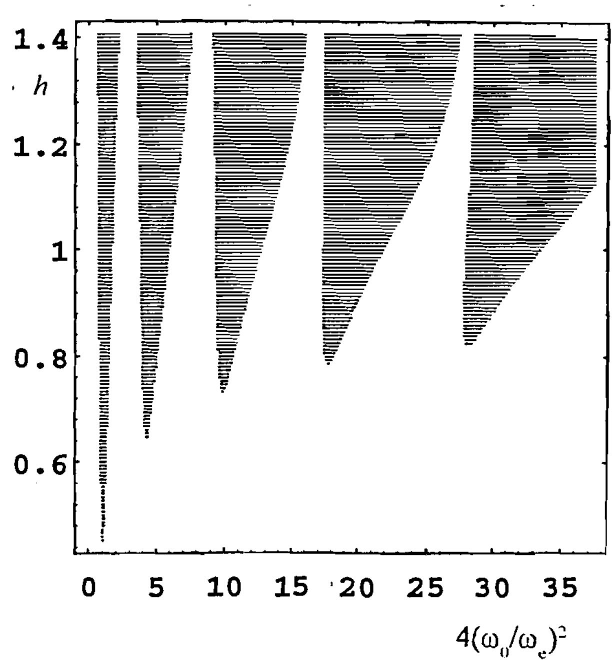
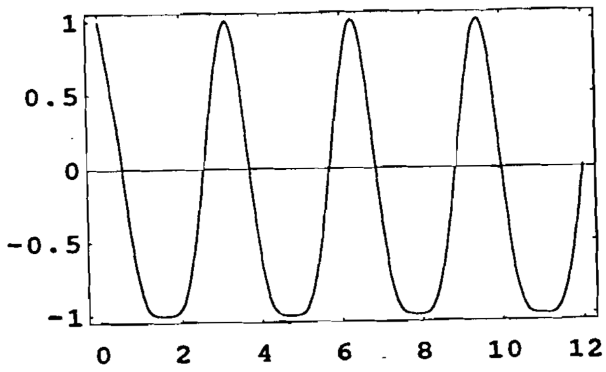


Fig. 4: The instability domains on the plane of h versus $4(\omega_0/\omega_e)^2$.



y=cos(2T) + 0.25cos(4T) - 0.25

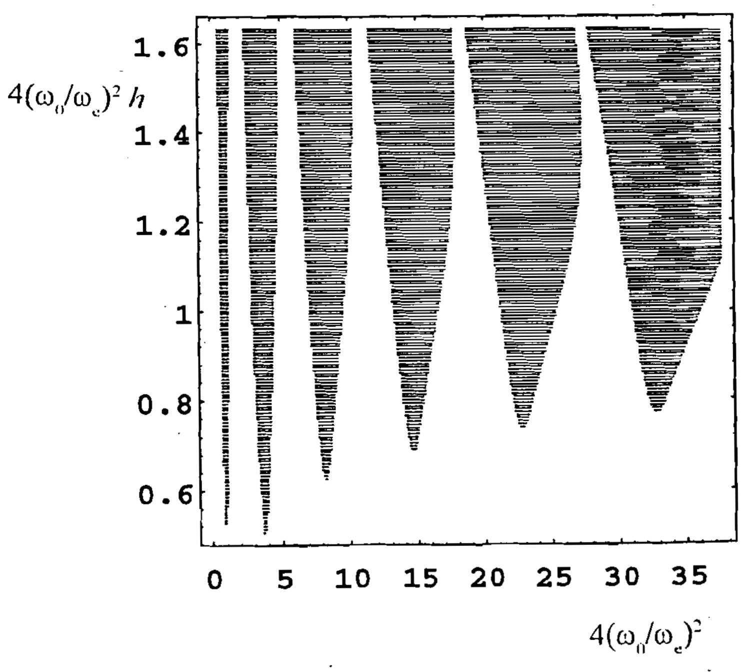


Fig 5: The effect of asymmetry