ON COURSE-STABILITY AND CONTROL DELAY

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Int. Shipbuild. Progr., **46**, no. 448 (1999) pp. 421-443

Received: April 1997
Accepted: August 1998

For many engineering systems that either rely on some sort of feedback for their stability or feature retardation in their responses, the type and magnitude of delay is critical for their behaviour. In this paper, a comprehensive treatment of the effect of delayed control on course-keeping capability is presented. Independent time-lags in the heading angle and yaw rate feedbacks are considered, in discrete as well as in continuous form. Stability boundaries are derived, either numerically or through approximate analytical solutions. Exact analytical solutions are sometimes possible. Two cases of behaviour in large following waves where the linear approach cannot be applied are also considered. The first concerns a parametrically driven system in yaw. The second is about oscillatory surf-riding.
Keywords: steering, manoeuvring, course-keeping, delay, stability, control.

1. Introduction

The first efforts for automatic steering of ships date back to studies based on the gyro-compass by Sperry (1915 & 1922). Noteworthy analytical and experimental investigations were first undertaken however by Minorsky (1922 & 1930). Minorsky was apparently the first to investigate also the effect of various transmission time-lags with a third-order functional differential equation of yaw. More intensive research followed the Second World War. Gimprich & Schiff (1949) have used the sway/yaw model of Davidson & Schiff (1946) and they described the delayed rudder response due to the inertia of the steering gear as a first-order linear differential equation of the rudder angle. This is equivalent with considering an exponential time-lag. For fast moving objects they discussed also the need of taking into account constant-type lags.
Chou et al. (1974), during their widely known radio-controlled model tests in San Fransisco Bay, experimented with an autopilot having a dead-band as a weather-adjustment instrument in rough seas and identical time-lags for yaw angle and rate. The

exponential -type response of the steering gear was taken into account by Åstrom (1980) in his design method for fixed gain and adaptive autopilots. More recently, Papoulias et al. (1994) examined the influence of positional information time-lags on track-keeping. In the SOLAS Convention is prescribed that the steering gear should be capable of putting the rudder from 35 deg on one side to +30 deg on the other side in less than 28 sec, when running at maximum service speed and with deepest seagoing draught. It would be perhaps more sensible however to have the rudder's rate scaled on the basis of ship size and speed, Eda & Crane (1965). Nowadays rudder response rates as high as 15-20 deg/s are possible, Fossen (1994).

The derivation of a universal stability criterion for dynamical systems with dependence on a past state has been pursued for many years. As a result, the general literature is abound with stability criteria. They are classified as, analytical, numerical, or, geometrical. Well known criteria are, *Pontryagin's,* that is quite popular in mathematics, the *τ-decomposition method*, often referenced in mechanics and *Nyquist's criterion* that is widely used by control engineers, Stépán (1989). Whilst it is generally true that delays tend to enlarge instability regions, in certain cases delays can turn unstable systems into stable. Also, the arrangement in the parameters' space of stable and unstable domains can be quite complicated, particularly if the delays are allowed to become large. An interesting example of an equation with cubic-type nonlinearity in restoring and time-lag in this term can be found in Kapitaniak, (1991). There is shown that increased delay causes decrement of amplitude at resonance while the unstable parts of the steady response curve near resonance tend to become stable. Nonlinear approaches for ship-specific delay problems are rather rare. Minorsky again, (1948), discussed the active roll stabilization problem on the basis of a second-order equation with retardation in damping. The interesting dynamics exhibited by this system were reconsidered recently by Mitsui et al. (1994).

In the present paper we consider the following aspects: Initially we analyze how independent "proportional" and "rate" time lags affect yaw stability. The accuracy of analytical approximations is assessed through comparison with numerical predictions. Exact analytical results are also presented when this is possible. Next we consider the problem of having continuous-type control delays. As shown this poses no serious problem if the weighting functions in the integrals representing the feedback are known and the system is still linear. In the remaining sections of the paper we investigate some aspects of the effect of long and steep regular waves for ship steering. Firstly, we examine a case of parametric instability in yaw that was recently pointed out, Spyrou, (1997). The first approximation of the effect of delay on the stability boundary that corresponds to the principal resonance is presented. The final problem that we analyze concerns the onset of oscillatory surf-riding, Spyrou (1995), based on a multi-degree nonlinear mathematical model.

2. General formulation

The stability of solutions of differential equations with delay is discussed in detail in several books, such as, Mishkis (1951), Minorsky (1962), Belman & Cook (1963), Hale (1977) and Stépán (1989). The simplest mathematical formulation corresponds to the so-called *difference-differential equations* (*"d.d.e"*). A linear second-order d.d.e with respect to a variable x (which could also be a vector) with, say, two independent delays in restoring, r_{11}, r_{12} and two other delays in damping, r_{21}, r_{22}, appears in the following form:

$$\ddot{x}(t) = -a_{21}\,\dot{x}(t - r_{21}) - a_{22}\,\dot{x}(t - r_{22}) - a_{11}\,x(t - r_{11}) - a_{12}\,x(t - r_{12}), \quad \text{for } r_{ij} > 0 \tag{1}$$

In (1) the dots indicate differentiation over time t and a_{ij}, $i = 1,2\ldots$, $j = 1,2\ldots$ are constants. Equation (1) is a special case of the more general *functional* representation:

$$\ddot{x}(t) = -\int_{-\tau_2}^{0} a_2(r)\,\dot{x}(t+r)\,dr - \int_{-\tau_1}^{0} a_1(r)\,x(t+r)\,dr \tag{2}$$

The terms $a_1(r)$ and $a_2(r)$ can be regarded as weighting factors, while τ_1 and τ_2 indicate delay intervals. To examine stability we generally proceed either by taking the Laplace transform of the two sides of (2), or more simply by substituting a solution $x = x_0 e^{\lambda t}$ in order to derive the corresponding *characteristic function*, $D(\lambda)$ which is then given by:

$$D(\lambda) = \lambda^2 + \lambda \int_{-\tau_2}^{0} e^{\lambda r} a_2(r)\,dr + \int_{-\tau_1}^{0} e^{\lambda r} a_1(r)\,dr \tag{3}$$

Stability depends on the roots of the above function which are the eigenvalues (the so-called *poles*) of our system. We are especially interested about the sign of their real parts since instability arises as soon as any of these turn positive. However, as λ appears also as an exponent in (3), the characteristic function is of transcendental nature and usually for such equations exact analytical expressions of the poles cannot be derived. In fact, for such a system an infinite number of eigenvalues exist, irrespectively of the order of the d.d.e. (this is easily seen if we think in terms of the series expansion of the exponential term which is an "infinite-order" polynomial of λ). Thus the difficulty about solving analytically a d.d.e lies before anything else in the linear formulation itself.

For several engineering systems that use feedback, either of discrete type or continuously sampled over a short period of time before present, it is sensible to treat delays as relatively small quantities and thus it is often sufficient to make use of a truncated Taylor-series expansion. The equation obtained can then be treated in the customary way for ordinary differential equations since the "lags" appear as external coefficients. In general however one has to bear in mind that some important dynamics may be eliminated during the approximation process. Numerical approaches based on a high-dimensional approximation of the system's infinite phase-space offer nowadays a much better potential, Mitsui et al. (1994).

3. Effect of discrete delays

Consider the simplified linear ship manoeuvring model of Nomoto (1972):

$$T'\ddot{\psi}' + \dot{\psi}' = K'\delta \tag{4}$$

where T', K' are the well known time and gain constants; and ψ, δ are the heading and rudder angles. For our notation where positive rudder angles result in positive yaw, T', K' are positive for a directionally stable ship. The primes indicate nondimensionalized quantities. Time is nondimensionalized on the basis of the relation $t' = t\frac{U}{L}$ where U, L are respectively, forward speed and ship length.

A control law based on proportional and differential gains is further introduced. Feedbacks are obtained at two different time instants before present for heading angle and at two other instants for yaw rate. In addition, an "exponential-type" response model of the steering gear is assumed:

$$\dot{\delta}' = \frac{1}{T_A'}\left[-\delta - k_{11}\,\psi(t' - r_{11}') - k_{12}\,\psi(t' - r_{12}') - k_{21}'\,\dot{\psi}(t' - r_{21}') - k_{22}'\,\dot{\psi}(t' - r_{22}')\right] \tag{5}$$

The new symbols that appear in (5) are, the rudder's time constant, T_A', the *proportional* to the heading angle gains, k_{11}, k_{12}, and the proportional to the yaw rate (*differential*) gains, k_{21}', k_{22}'.

By combining (4) with (5) we can eliminate δ and $\dot{\delta}'$:

$$T'\psi'^{(3)} + \left(1 + \frac{T'}{T_A'}\right)\ddot{\psi}' + \frac{1}{T_A'}\dot{\psi}' + \frac{k_{11}K'}{T_A'}\psi(t' - r_{11}') + \frac{k_{12}K'}{T_A'}\psi(t' - r_{12}') + \\ + \frac{k_{21}'K'}{T_A'}\dot{\psi}'(t' - r_{21}') + \frac{k_{22}'K'}{T_A'}\dot{\psi}'(t' - r_{22}') = 0 \tag{6}$$

Characteristic function:

$$D(\lambda) = T'\lambda^3 + \left(1 + \frac{T'}{T'_A}\right)\lambda^2 + \frac{1}{T'_A}\lambda + \frac{k_{11}K'}{T'_A}e^{-r'_{11}\lambda} + \frac{k_{12}K'}{T'_A}e^{-r'_{12}\lambda} + \\ + \frac{k'_{21}K'}{T'_A}\lambda\, e^{-r'_{21}\lambda} + \frac{k'_{22}K'}{T'_A}\lambda\, e^{-r'_{22}\lambda} \tag{7}$$

The poles of (7) can be real or complex. Here however we are interested only to know where the imaginary axis of the complex plane is crossed because this implies change of stability. Since we are looking for poles with zero real-part we can substitute $\lambda = i\,y$ in (7) (therefore y is real) and then request the real and imaginary parts of the characteristic function $D(\lambda)$, respectively $R(y)$ and $I(y)$, to become simultaneously equal to zero:

$$R(y) = -\left(1 + \frac{T'}{T'_A}\right)y^2 + \frac{k_{11}K'}{T'_A}\cos(r'_{11}y) + \frac{k_{12}K'}{T'_A}\cos(r'_{12}y) + \\ + \frac{k'_{21}K'}{T'_A}y\sin(r'_{21}y) + \frac{k'_{22}K'}{T'_A}y\sin(r'_{22}y) = 0 \tag{8}$$

$$I(y) = -T'y^3 + \frac{1}{T'_A}y - \frac{k_{11}K'}{T'_A}\sin(r'_{11}y) - \frac{k_{12}K'}{T'_A}\sin(r'_{12}y) + \\ + \frac{k'_{21}K'}{T'_A}y\cos(r'_{21}y) + \frac{k'_{22}K'}{T'_A}y\cos(r'_{22}y) = 0 \tag{9}$$

Equations (8) and (9) are transcendental and, in principle, they cannot be solved analytically. The alternatives are, an approximation through a Taylor-series expansion or a numerical scheme. The latter will be more accurate but cannot produce solutions in the desirable closed form. If the delay terms are small, we can use the first-order approximation of (6):

$$\psi'^{(3)} + \underbrace{\left[\frac{1}{T'} + \frac{1}{T'_A} + \frac{K'}{T'}\frac{1}{T'_A}(k'_{21}r'_{21} + k'_{22}r'_{22})\right]}_{A}\ddot{\psi}' + \\ + \underbrace{\left[\frac{1}{T'T'_A} - \frac{K'}{T'}\frac{1}{T'_A}(k_{11}r'_{11} + k_{12}r'_{12}) + \frac{K'}{T'}\frac{1}{T'_A}(k'_{21} + k'_{22})\right]}_{B}\dot{\psi}' + \underbrace{\frac{K'}{T'}\frac{1}{T'_A}(k_{11} + k_{12})}_{C}\psi = 0 \tag{10}$$

or,

$$\psi'^{(3)} + A\ddot{\psi}' + B\dot{\psi}' + C\psi = 0 \tag{10'}$$

According to the Ruth-Hurwitz criteria the conditions for stability are:
$AB > C$, $A > 0$, $B > 0$, $C > 0$. Given that $\frac{K'}{T'}$, T_A' and the gains are positive, the last inequality is always true. If we concentrate on a case where the unsteered ship is directionally stable, there will be $T' > 0$. Combined with the fact that the delays are assumed positive, the condition $A > 0$ is also satisfied. The condition $B > 0$ leads to the simple relation $\frac{1}{K'} - (k_{11}r_{11}' + k_{12}r_{12}') + k_{21}' + k_{22}' > 0$. Finally the first inequality provides the condition:

$$\left[\frac{1}{T'} + \frac{1}{T_A'} + \frac{K'}{T'}\frac{1}{T_A'}(k_{21}' r_{21}' + k_{22}' r_{22}')\right]\left[\frac{1}{T'T_A'} - \frac{K'}{T'}\frac{1}{T_A'}(k_{11}' r_{11}' + k_{12}' r_{12}') + \frac{K'}{T'}\frac{1}{T_A'}(k_{21}' + k_{22}')\right] > \frac{K'}{T'}\frac{1}{T_A'}(k_{11} + k_{12}) \tag{11}$$

which, since $\frac{K'}{T'}$, $T_A' > 0$ is equivalent with:

$$\left[\frac{1}{T'} + \frac{1}{T_A'} + \frac{K'}{T'}\frac{1}{T_A'}(k_{21}' r_{21}' + k_{22}' r_{22}')\right]\left[\frac{1}{K'} - (k_{11} r_{11}' + k_{12} r_{12}') + (k_{21}' + k_{22}')\right] > (k_{11} + k_{12}) \tag{11'}$$

It is obvious that because (11') is more stringent than the condition $A > 0 \Leftrightarrow \frac{1}{K'} - (k_{11}r_{11}' + k_{12}r_{12}') + k_{21}' + k_{22}' > 0$ the latter is redundant. Therefore (11) is the key condition for identifying the critical for stability gain-delay combinations.
For $k_{21}' = k_{22}' = 0$ (11') becomes :

$$k_{11} r_{11}' + k_{12} r_{12}' < \frac{1}{K'} - \frac{(k_{11} + k_{12})T'T_A'}{(T' + T_A')} \tag{12}$$

By following a similar procedure it can be shown that the condition that corresponds to the third-order approximation is :

$$-\frac{K'^2}{6}\left(k_{11}+k_{12}\right)\left(k_{11}\,r_{11}'^{\,3}+k_{12}\,r_{12}'^{\,3}\right)+\frac{K'^2}{2}\left(k_{11}\,r_{11}'^{\,2}+k_{12}\,r_{12}'^{\,2}\right)\left(k_{11}\,r_{11}'+k_{12}\,r_{12}'\right)-$$

$$-\frac{K'}{2}\left(k_{11}\,r_{11}'^{\,2}+k_{12}\,r_{12}'^{\,2}\right)+K'\left(T'+T_A'\right)\left(k_{11}\,r_{11}'+k_{12}\,r_{12}'\right)\;<\qquad(13)$$

$$<\;\left(T'+T_A'\right)-K'\,T'\,T_A'\left(k_{11}+k_{12}\right)$$

The second-order approximation can be obtained simply by neglecting the third order terms of (13). Mean curves of the relation between K' and T' based on regression analysis from a large number of ships can be found in Barr et al. (1981): $K'=0.625+0.375\,T'$, and also in ITTC (1987): $K'=0.452+0.481\,T'$. In Figure 1 are compared the first and third-order approximations on the plane $\left(r_{11}', r_{12}'\right)$ for $K'=1.5$, $T'=2.0$, $k_{11}=k_{12}=1$, $k_{21}'=k_{22}'=0$ and for three different values of the time constant T_A' $(=0.1,\ 0.2$ and $0.3)$ which accounts for another type of delay, resulting from the retardation in the steering gear. If T_A' becomes large the permissible r_{ij}' values should be reduced in order to keep the "total delay" confined. Schiff & Gimprich (1949) had considered for T_A' a value about 0.1. The system of (8), (9) was solved also numerically. These solutions were found essentially coinciding with the third-order approximations. In some special cases we can eliminate the sinusoidal terms from the pair of equations (8) and (9) and proceed analytically. Two such cases are considered below in parallel: (a) when $k_{21}'=k_{22}'=0$ (only proportional delays); and (b) when $k_{12}'=k_{21}'=0$ (one proportional and one rate delay). Then, by shifting the sinusoidal terms of (8) and (9) to the right-hand-side, raising to second-order and adding we obtain respectively the following equations:

$$\frac{T'^2}{K'^2}y^6+\frac{T'^2+T_A'^2}{K'^2\,T_A'^2}y^4+\frac{1}{K'^2\,T_A'^2}y^2-\frac{2k_{11}k_{12}}{T_A'^2}\cos\left[\left(r_{11}'-r_{12}'\right)y\right]-\frac{k_{11}^{\,2}+k_{12}^{\,2}}{T_A'^2}=0\qquad(14a)$$

$$\frac{T'^2}{K'^2}y^6+\frac{T'^2+T_A'^2}{K'^2\,T_A'^2}y^4+\frac{1-k_{22}'^{\,2}\,K'^2}{K'^2\,T_A'^2}y^2-\frac{2k_{11}k_{22}'}{T_A'^2}y\,\sin\left[\left(r_{22}'-r_{11}'\right)y\right]-\frac{k_{11}^{\,2}}{T_A'^2}=0\qquad(14b)$$

The above can be solved analytically for y only when the delays are identical because then they both reduce into third-order polynomials of y^2:

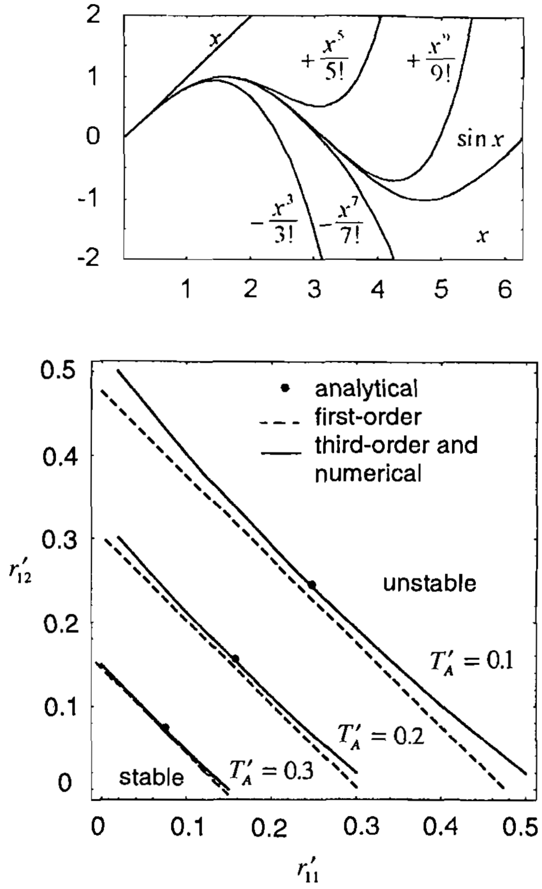

Figure 1. In the lower figure are shown the stability boundaries when two independent proportional delays, r_{11}' and r_{12}', exist. Three different values of the rudder's time constant, T_A', are considered. Analytical (1st order, 3rd order and exact) and numerical solutions are presented. The 3rd order solution practically coincides with the numerical. The required order of the series depends in general on the lag magnitude and the considered function type. The upper figure shows the required number of terms for approximation of the sine function for different intervals of the argument.

$$\frac{T'^2}{K'^2}\left(y^2\right)^3+\frac{T'^2+T_A'^2}{K'^2\,T_A'^2}\left(y^2\right)^2+\frac{1}{K'^2\,T_A'^2}\left(y^2\right)-\frac{\left(k_{11}+k_{12}\right)^2}{T_A'^2}=0 \tag{15a}$$

$$\frac{T'^2}{K'^2}\left(y^2\right)^3+\frac{T'^2+T_A'^2}{K'^2\,T_A'^2}\left(y^2\right)^2+\frac{1-k_{22}'^2\,K'^2}{K'^2\,T_A'^2}\left(y^2\right)-\frac{k_{11}^2}{T_A'^2}=0 \tag{15b}$$

The exact solutions of (15a) and (15b) can be found analytically with *Mathematica* (their detailed forms are omitted due to their length). In Figure 1 the acceptable analytical solution of (15a) is compared with the approximate solution found earlier and also with a numerical solution.

If is assumed that the rudder obtains the desired angle almost instantaneously, that means $T_A' \approx 0$, then (15a) and (15b) can be simplified further since they reduce into the following quadratics in terms of y^2:

$$\frac{T'^2}{K'^2}\left(y^2\right)^2+\frac{1}{K'^2}\left(y^2\right)-\left(k_{11}+k_{12}\right)^2=0 \tag{16a}$$

$$\frac{T'^2}{K'^2}\left(y^2\right)^2+\frac{1-k_{22}'\,K'^2}{K'^2}\left(y^2\right)-k_{11}^2=0 \tag{16b}$$

Equation (16a) and (16b) have two real and two imaginary roots. As explained in section 3 we are interested only in the real roots which are given by:

$$y_{1,2}=\pm\frac{1}{T'}\sqrt{\frac{-1+\sqrt{1+4K'^2\,T'^2\left(k_{11}+k_{12}\right)^2}}{2}} \tag{17a}$$

$$y_{1,2}=\pm\frac{1}{T'}\sqrt{\frac{-\left(1-k_{22}'^2\right)+\sqrt{\left(1-k_{22}'^2\right)^2+4K'^2\,T'^2\,k_{11}^2}}{2}} \tag{17b}$$

With substitution of y_1 or y_2 into the simplified version of (9) we derive the stability boundary. For example for system (a) with two independent proportional gains this is given by the following equality (see also Figure 2):

$$\frac{1}{T'}\sqrt{\frac{-1+\sqrt{1+4K'^2\,T'^2\left(k_{11}+k_{12}\right)^2}}{2}}= \\ = K'\left(k_{11}+k_{12}\right)\sin\left(r_{11}'\frac{1}{T'}\sqrt{\frac{-1+\sqrt{1+4K'^2\,T'^2\left(k_{11}+k_{12}\right)^2}}{2}}\right) \tag{18}$$

The parametric equations of the poles of the full Nomoto's equation are derived in the Appendix.

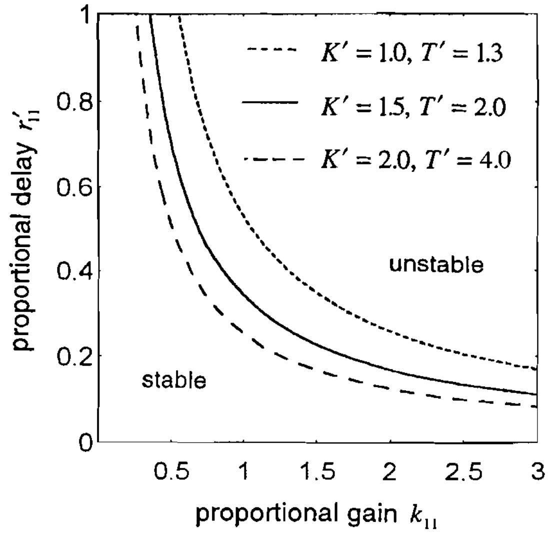

Figure 2. Stability boundaries on the plane of proportional gain k_{11} versus the delay magnitude r_{11}', for three different pairs of K' and T'.

4. Continuous delay

Eq. (4) will be combined now with an autopilot equation featuring finite continuous time-lags in the yaw angle and rate feedback terms:

$$\delta = -k_1 \int_{-1}^{0} w_1(r')\psi(t'+r')dr' - k_2' \int_{-1}^{0} w_2(r')\dot{\psi}'(t'+r')dr' \tag{19}$$

The weighting functions $w_1(r')$ and $w_2(r')$ are scaled for the delay interval so that they satisfy the relation $\int_{-1}^{0} w_i(r')dr' = 1,\ i = 1,2$. The combination of (4) and (19) yields:

$$\ddot{\psi}' + \frac{1}{T'}\dot{\psi}' + k_2'\frac{K'}{T'}\int_{-1}^{0} w_2(r')\dot{\psi}'(t'+r')dr' + k_1\frac{K'}{T'}\int_{-1}^{0} w_1(r')\psi(t'+r')dr' = 0 \tag{20}$$

For simplicity will be assumed that $w_1(r') = w_2(r') = w(r')$. Let's select for demonstration purposes a weighting function given by $w(r') = \frac{\pi}{2}\cos\left(\frac{\pi r'}{2}\right)$. After calculation of a number of integrals, the characteristic function of (20) is expressed as:

$$D(\lambda) = \lambda^2 + \frac{1}{T'}\lambda + k_2' \frac{K'}{T'} \frac{\lambda\left(2\lambda\pi + \pi^2 e^{-\lambda}\right)}{4\lambda^2 + \pi^2} + k_1 \frac{K'}{T'} \frac{\left(2\lambda\pi + \pi^2 e^{-\lambda}\right)}{4\lambda^2 + \pi^2} \tag{21}$$

By substituting $\lambda = i\,y$ and setting the real and imaginary parts equal to zero, we obtain:

$$R(y) = -y^2 - k_2' \frac{K'}{T'} \frac{2\pi y^2}{\pi^2 - 4y^2} + k_2' \frac{K'}{T'} \frac{\pi^2 \sin y}{\pi^2 - 4y^2} + k_1 \frac{K'}{T'} \frac{\pi^2 \cos y}{\pi^2 - 4y^2} = 0 \tag{22}$$

$$I(y) = \frac{1}{T'} y + k_2' \frac{K'}{T'} \frac{\pi^2 y \cos y}{\pi^2 - 4y^2} + k_1 \frac{K'}{T'} \frac{2\pi y}{\pi^2 - 4y^2} - k_1 \frac{K'}{T'} \frac{\pi^2 \sin y}{\pi^2 - 4y^2} = 0 \tag{23}$$

The system of (22) and (23) was solved numerically. The solution, for $K' = 1.5$, $T' = 2.0$ and $k_1 = k_2' = 1$, is shown in Figure 3 (the arrows point towards the area of stability). The weighting functions can be generalized by considering a family with the form $w(r') = n\frac{\pi}{2}\cos\left(n\frac{\pi r'}{2}\right)$ each member of which is associated with the delay interval $\left[-\frac{1}{n}, 0\right]$ where n is a positive integer. Thus by increasing n the delay interval becomes smaller (see the inserted graph of Figure 3). As all these functions satisfy automatically the condition $\int_{-\frac{1}{n}}^{0} w(r')dr' = 1$ the area is always equal to 1.0. The stability boundaries that correspond to $n = 2$ and $n = 3$ can also be seen in Figure 3.

5. Delay and parametric instability

Let's introduce now at the right-hand-side of (4) a term playing the role of sinusoidal wave excitation in following waves, Spyrou (1997):

$$T'\ddot{\psi}' + \dot{\psi}' = K'\delta + A'\psi\cos(\omega_e' t') \tag{24}$$

Above, A' is the amplitude of wave excitation and ω_e' is the encounter frequency. Consider rudder control as follows :

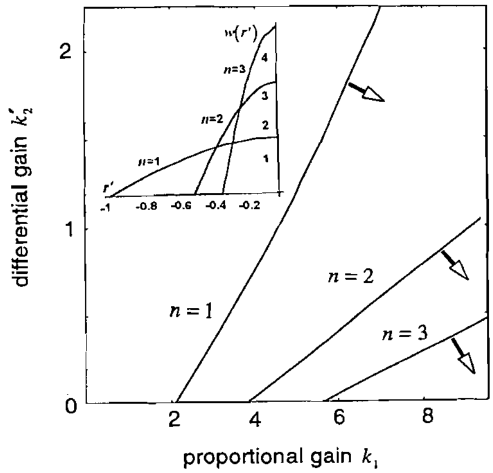

Figure 3. Stability boundaries for "continuous-type" delay, based on a weighting function given by the expression $w(r') = (n\frac{\pi}{2})\cos(nr'\frac{\pi}{2})$, with $r' \in [-1,0]$..

$$\delta = -k_1\left[\psi(t'-r_1') - \psi_r\right] - k_2'\dot{\psi}'(t'-r_2') \tag{25}$$

The parameter ψ_r represents the desired heading of the ship. Substitution of (25) into (24) yields:

$$\ddot{\psi}' + \frac{1}{T'}\dot{\psi}' + k_2'\frac{K'}{T'}\dot{\psi}'(t'-r_2') + k_1'\frac{K'}{T'}\psi(t'-r_1') - \frac{A'}{T'}\psi\cos(\omega_e' t') = k_1\frac{K'}{T'}\psi_r \tag{26}$$

The version of (26) without delay has been shown to be Mathieu-type, with simultaneous parametric and constant direct forcing, Spyrou (1997):

$$\ddot{\psi}' + \beta\dot{\psi}' + \omega_0'^2\left[1 - h\cos(\omega_e' t')\right]\psi = j \tag{27}$$

The quantity ω_0' represents the natural frequency of the automatically controlled ship in yaw and is given by the relation $\omega_0' = \sqrt{k_1\frac{K'}{T'}}$. Furthermore, $\beta = \frac{1+k_2'K'}{T'}$ is the damping, $h = \frac{A'}{k_1K'}$ is the amplitude of parametric forcing, and $j = k_1\frac{K'}{T'}\psi_r$ is the constant independent forcing (which, given the wave, depends on the desired heading, ψ_r).

Given that in practice these delays are not expected to be large quantities, it is possible to develop some understanding of how they tend to influence behaviour by applying the following first-order approximations:

$$\psi(t'-r_1') \approx \psi(t') - r_1'\dot{\psi}'(t')$$

$$\dot{\psi}(t'-r_2') \approx \dot{\psi}(t') - r_2'\ddot{\psi}'(t') \tag{28}$$

Substitution into (26) yields:

$$\left(1 - r_2' k_2' \frac{K'}{T'}\right)\ddot{\psi}' + \left(\frac{1}{T'} - r_1' k_1 \frac{K'}{T'} + k_2' \frac{K'}{T'}\right)\dot{\psi}' + \left[k_1 \frac{K'}{T'} - \frac{A'}{T'}\cos(\omega_e' t')\right]\psi = k_1 \frac{K'}{T'}\psi_r \tag{29}$$

From a comparison of (27) and (29) the following relationships are deduced:

$$\omega_0'^{del(1)} = \frac{\omega_0'}{\sqrt{1 - r_2' \frac{k_2'}{k_1}\omega_0'^2}}, \qquad \beta^{del(1)} = \frac{\beta - r_1'\omega_0'^2}{1 - r_2' \frac{k_2'}{k_1}\omega_0'^2} \tag{30}$$

where $\omega_0'^{del(1)}$ and $\beta^{del(1)}$ are respectively, yaw natural frequency and damping of the delayed system at first approximation. It is reasonable to assume that $1 - r_2' \frac{k_2'}{k_1}\omega_0'^2 > 0$ since r_2' was considered as small quantity, k_2' is normally about 1.0 while an average value of $\frac{1}{2}\frac{K'}{T'}$ (that is the turning index P of Norrbin) can be taken to be about o.5.
According to the classical analysis of Mathieu's equation, for $j = 0$ (that means basically $\psi_r = 0$), the boundary of the first instability region (principal resonance), when damping can be treated as a small quantity, is obtained from the condition: $h_{\min} = \frac{2\beta}{\omega_0'}$. The minimal amplitude of parametric forcing for the delayed system, $h_{\min}{}^{del(1)}$ will be at first order:

$$h_{\min}{}^{del(1)} = \frac{2\beta^{del(1)}}{\omega_0'^{del(1)}} \text{ or, } h_{\min}{}^{del(1)} = \frac{2\beta}{\omega_0'}\frac{1}{\sqrt{1 - r_2' \frac{k_2'}{k_1}\omega_0'^2}} - \frac{2r_1'\omega_0'}{\sqrt{1 - r_2' \frac{k_2'}{k_1}\omega_0'^2}} \text{ or,}$$

$$h_{\min}{}^{del(1)} = \frac{1}{\sqrt{1 - r_2' \frac{k_2'}{k_1}\omega_0'^2}}\left(h_{\min} - 2r_1'\omega_0'\right) \tag{31}$$

where $h_{\min}$ is the minimum amplitude required to destabilize the system when there is no delay. From (31) is deduced that the relation between $h_{\min}$ and $h_{\min}{}^{del}$ can be approximated with a straight-line. Such plots are shown in Figure 4 for $\frac{K'}{T'} = 0.75$, $k_1 = k_2' = 1$ and various values of r_1' and r_2'. As can be seen, when $r_1' = 0$ some small rate delay $r_2' > 0$ can in fact lift the limit of allowable parametric excitation before instability is incurred. However if both delays exist the allowable excitation is much smaller. Thus the presence of the proportional delay clearly enlarges the instability region. On the other hand it can be derived from (31) that the effect of the differential delay is beneficial (under the condition $1 - r_2' \frac{k_2'}{k_1} \omega_0'^2 > 0$) if:

$$r_2' < \frac{1 - \left(1 - \frac{2 r_1' \omega_0'}{h_{\min}}\right)^2}{\frac{k_2'}{k_1} \omega_0'^2} \tag{32}$$

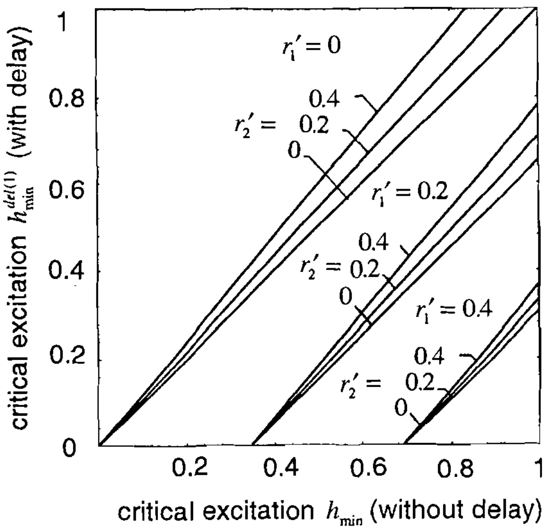

Figure 4. Effect of delay on the limiting amplitude of parametric excitation required for instability in the region of principal resonance. When the proportional delay r_1' is zero some small differential delay r_2' can lift the minimal allowed excitation. When both delays are present the allowed excitation is much lower compared to that of a system with no delay.

If a second-order approximation is considered, then the differential equation becomes a third-order equation of ψ due to the presence of the $r_2'^2$ term that multiplies $\psi'^{(3)}$. This is undesirable because we can no longer gain insights on the basis of the well known properties of Mathieu's equation. If we discard r_2' and proceed only with the first delay, r_1', we obtain:

$$\omega_0'^{del(2)} = \frac{\omega_0'}{\sqrt{1+r_1'^2\omega_0'^2}},\ \beta^{del(2)} = \frac{\beta - r_1'\omega_0'^2}{1+r_1'^2\omega_0'^2},\ h_{\min}{}^{del(2)} = \frac{1}{\sqrt{1+r_1'^2\omega_0'^2}}\left(h_{\min} - 2r_1'\omega_0'\right) \quad (33)$$

In Figure 5 is shown the dependence of $h_{\min}{}^{del}$ on $\frac{K'}{T'}$ with fixed $h_{\min}$. In the $\frac{K'}{T'}$ range [0, 15] the difference between first and second-order approximations is hardly noticeable.

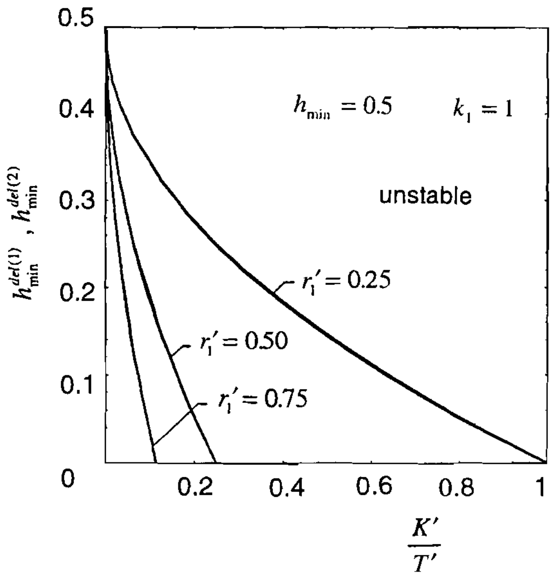

Figure 5. Dependence of the critical amplitude of parametric excitation with $\frac{K'}{T'}$. *The magnitude of proportional delay is varied. The first and second-order solutions are practically indistinguishable for the considered range.*

6. Time-lags as bifurcation parameters of multi-degree and nonlinear systems

The final case to examine will be about the effect of delayed control on the rather complicated dynamics of a small fishing vessel with four degrees of freedom (surge-sway-yaw and roll). This ship can exhibit in steep waves very rich dynamic behaviour, including several bifurcations and even giving rise to a short chaotic regime, Spyrou

(1995 & 1996). We shall examine how different magnitudes of delay in the proportional or differential terms of the autopilot equation influence the onset of self-sustained oscillations which arise through a Hopf bifurcation, when the ship is in surf-riding condition.

Undesirable time-lags are often behind self-sustained oscillatory behaviour in several dynamical systems. Also, it is reasonable to expect that as the delays increase in magnitude the oscillations will tend to appear at lower excitation levels. Once more, the delays ought to be in a practical sense low and we shall proceed with first-order approximations. The autopilot equation is:

$$\dot{\delta}' = \frac{1}{T_A'}\left\{-\delta - k_1\left[\psi(t' - r_1') - \psi_r\right] - k_2' \dot{\psi}'(t' - r_2')\right\} \tag{34}$$

First-order approximation :

$$\dot{\delta}' = \frac{1}{T_A'}\left\{-\delta - k_1\left[\psi - \psi_r\right] - (k_2' - k_1 r')\dot{\psi}' + k_2' r_2' \ddot{\psi}\right\} \tag{35}$$

If we set $k_2 = k_2' - k_1 r_1'$ and $k_3 = -k_2' r_2'$, then:

$$\dot{\delta}' = \frac{1}{T_A'}\left[-\delta - k_1(\psi - \psi_r) - k_2 \dot{\psi}' - k_3 \ddot{\psi}'\right] \tag{36}$$

It becomes apparent from (36) that, at first order, the existence of delay in the proportional term appears as a reduction of the differential gain. Similarly, the presence of delay in the differential term plays the role of negative acceleration gain which makes the autopilot less capable in dealing with rapid yawing motions. As an application we selected the following dimensional values for the autopilot parameters: $T_A = 1.0$ s/rad, $k_1 = 1.0$ rad and $k_2 = 1.0$ rad/s. The wave characteristics are, $\frac{\lambda}{L} = 2.0$ and $\frac{H}{\lambda} = \frac{1}{20}$.

At first the delay of the differential term was fixed at $r_2 = 0.6$s and the proportional delay r_1 was varied. For each r_1 the corresponding stationary state was numerically identified. Furthermore, for each state stability analysis was automatically carried out, based on numerical local linearization and calculation of eigenvalues. The local dynamics are to a great extent governed by the behaviour of a critical conjugate pair of eigenvalues whose real part is near to 0 and thus "controls" the onset of the Hopf bifurcation. This real part has been plotted in Figure 6 against the desired heading ψ_r. Clearly, by increasing r_1 the Hopf bifurcation arises at lower ψ_r. A similar trend is observed when r_1 is fixed and r_2 is varied, Figure 7. Interestingly, the curve obtained for $r_1 = 0.8$s and $r_2 = 0.9$s crosses twice the zero level and in the near-zero region of the control parameter ψ_r the considered eigenvalue-real-part turns positive.

In Figure 8 is shown the locus of the Hopf bifurcation which is also the threshold of self-sustained oscillatory behaviour. The Hopf bifurcation can arise at low desired headings if the delays are large enough. Detailed analysis in this area entails however higher-order approximations.

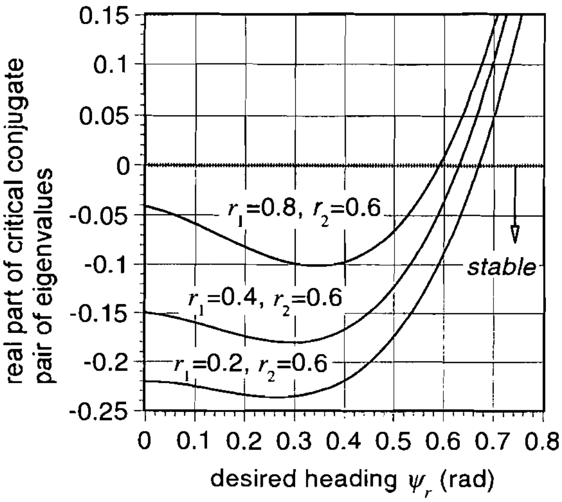

Figure 6. With increase of the proportional lag r_1 the critical pair of eigenvalues that govern the onset of instability tend to acquire positive real part at a progressively lower desired heading.

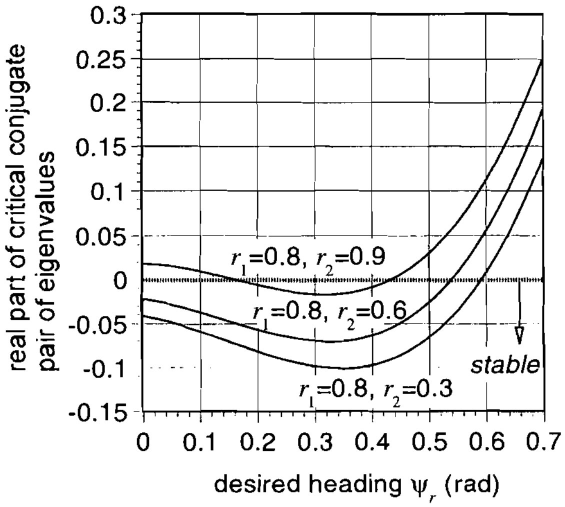

Figure 7. Similar investigation as in Figure 6, this time however with the proportional delay r_1 fixed and the "rate" delay r_2 varied. At relatively high values of r_2 there is double crossing of the zero axis. This means that Hopf bifurcations arise at two different values of desired heading.

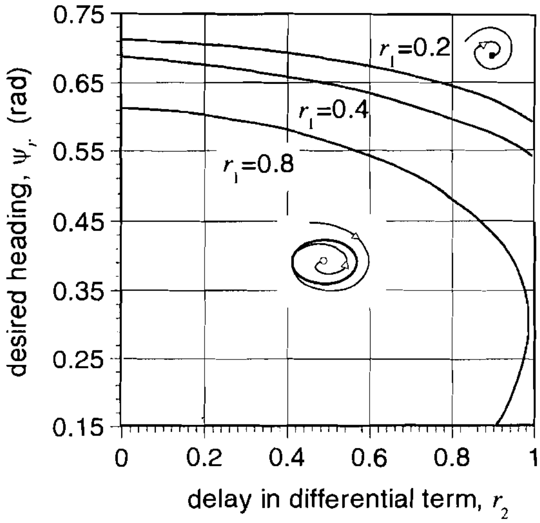

Figure 8. The boundary of self-sustained oscillatory behaviour, on the plane of "rate" delay r_2 versus the desired heading ψ_r.

7. A note on the effect of phase lead

When advanced actions are considered in the control law, the r_{ij} terms of (1) will be negative. It is interesting that sometimes we obtain a characteristic function which is essentially identical with this of a delayed system. A simple example is a second-order d.d.e with time-lag r both in damping and in restoring, Minorsky (1962):

$$\ddot{x} + a\dot{x}(t-r) + bx(t-r) = 0,\; r > 0 \tag{37}$$

The corresponding characteristic equation is $\lambda^2 + \lambda a e^{-r\lambda} + b e^{-r\lambda} = 0$ which can be written also as $\lambda^2 e^{r\lambda} + \lambda a + b = 0$. However the last expression is basically the characteristic function of a system with an "advanced inertia" :

$$\ddot{x}(t+r) + a\dot{x} + bx = 0 \tag{38}$$

8. Concluding remarks

The effect of various types of discrete and continuous delay were considered in the feedback loop of a "basic" course-keeping system. Expressions for the amount of admissible proportional and rate delays of constant type, as functions of the K', T' quantities of a ship, the gain values and the rudder's time constant (which is also a delay, however of exponential type) were presented. The higher the gains the lower the tolerable delays. For a ship with $K' = 1.5$, $T' = 2.0$, unit proportional gain, rudder time constant $T'_A = 0.1$, and two independent proportional delays r'_{11}, r'_{12}, stability exists if the approximate relationship $r'^2_{11} + r'^2_{12} \leq 0.5^2$ is satisfied (Figure 1).

The effect of continuous delay was examined with a sinusoidal-type weighting function. Due to the fact that the general approach does not distinguish between lags in the control system and retardations in the actual response terms, this same approach can be very effective also in handling hydrodynamic memory-effect problems. This area of research is currently under consideration.

The effect of delay in steering control when large following/quartering waves are present was also examined for two specific cases. The first case was a mechanism of parametric instability of yaw, resulting from the combined effect of the wave and the autopilot. First-order approximations of the relation linking the amplitude of parametric excitation with the delays were presented for the stability boundary of the principal resonance. If there is no proportional gain, some small delay may in fact beneficial for stability. If however both delays exist the allowable excitation is much lower.

Finally the problem of surf-riding in quartering waves with delayed control was investigated. Self-sustained oscillations arise through a Hopf bifurcation at a critical desired heading. The oscillation threshold was correlated with the magnitudes of proportional and rate delay. Increased delays shift this threshold at lower desired headings. Notably, for large delays oscillations are possible even at exactly following waves.

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Appendix

Let's consider the "full version" of Nomoto's linear manoeuvring model:

$$T_1'T_2'\,\psi'^{(3)} + \left(T_1' + T_2'\right)\ddot{\psi}' + \dot{\psi}' = K'\delta + K'T_3'\dot{\delta}' \tag{I1}$$

Consider also a control law with delays in the proportional and rate terms:

$$\dot{\delta}' = \frac{1}{T_A'}\left[-\delta - k_1\,\psi\left(t' - r_1'\right) - k_2'\,\dot{\psi}\left(t' - r_2'\right)\right] \tag{I2}$$

After elimination of δ and $\dot{\delta}'$ from (I1) and (I2) we obtain:

$$\frac{T_A'\,T_3'}{T_A' - T_3'}\psi'^{(4)} - \left[\frac{T_A'\,T_3'}{T_A' - T_3'}\left(\frac{1}{T_1'} + \frac{1}{T_2'}\right) + \frac{T_3'}{T_A' - T_3'}\right]\psi'^{(3)} +$$

$$+\left[\frac{T_3'}{T_A' - T_3'}\left(\frac{1}{T_1'} + \frac{1}{T_2'}\right) + \frac{T_A'\,T_3'}{T_1'\,T_2'\left(T_A' - T_3'\right)}\right]\ddot{\psi}' + \frac{T_3'}{T_1'T_2'\left(T_A' - T_3\right)}\dot{\psi}' +$$

$$+\frac{k_2'\,K'\,T_3'^2}{T_1'T_2'\left(T_A' - T_3'\right)}\ddot{\psi}\left(t' - r_2'\right) + \frac{k_1\,K'\,T_3'^2}{T_1'T_2'\left(T_A' - T_3'\right)}\dot{\psi}\left(t' - r_1'\right) + \tag{I3}$$

$$+\frac{k_2'\,K'\,T_3'}{T_1'T_2'\left(T_A' - T_3'\right)}\dot{\psi}\left(t' - r_2'\right) + \frac{k_1\,K'\,T_3'}{T_1'T_2'\left(T_A' - T_3'\right)}\psi\left(t' - r_1'\right) = 0$$

Proceeding as for the lower-order system of section 3, we first derive the characteristic function $D(\lambda)$ that corresponds to (I3). Then, by substituting $\lambda = iy$ and requesting the real and imaginary parts of $D(\lambda)$ to be equal to zero, we obtain finally the following pair of equations:

$$R(y)=\frac{T_A' T_3'}{T_A'-T_3'}y^4-\left[\frac{T_3'}{(T_A'-T_3')}\left(\frac{1}{T_1'}+\frac{1}{T_2'}\right)+\frac{T_A' T_3'}{T_1' T_2'(T_A'-T_3')}-\right]y^2-$$

$$-\frac{k_2' K' T_3'^2}{T_1' T_2'(T_A'-T_3')}y^2\cos(r_2' y)+\frac{k_1 K' T_3'^2}{T_1' T_2'(T_A'-T_3')}y\sin(r_1' y)+ \tag{I4}$$

$$+\frac{k_2' K' T_3'}{T_1' T_2'(T_A'-T_3')}y\sin(r_1' y)+\frac{k_1' K' T_3'}{T_1' T_2'(T_A'-T_3')}\cos(r_2' y)=0$$

$$I(y)=-\left[\frac{T_A' T_3'}{(T_A'-T_3')}\left(\frac{1}{T_1'}+\frac{1}{T_2'}\right)+\frac{T_3'}{T_A'-T_3'}\right]y^3+\frac{T_3'}{T_1' T_2'(T_A'-T_3')}y+$$

$$+\frac{k_2' K' T_3'^2}{T_1' T_2'(T_A'-T_3')}y^2\sin(r_2' y)+\frac{k_1 K' T_3'^2}{T_1' T_2'(T_A'-T_3')}y\cos(r_1' y)+ \tag{I5}$$

$$+\frac{k_2' K' T_3'}{T_1' T_2'(T_A'-T_3')}y\cos(r_2' y)-\frac{k_1' K' T_3'}{T_1' T_2'(T_A'-T_3')}\sin(r_1' y)=0$$

Generally, the roots of the two equations (I4) and (I5) cannot be found analytically. If T_A' is approximately zero, we obtain the simpler system :

$$R(y)=\left(\frac{1}{T_1'}+\frac{1}{T_2'}\right)y^2+\frac{k_2' K' T_3'}{T_1' T_2'}y^2\cos(r_2' y)-\left(\frac{k_1 K' T_3'}{T_1' T_2'}y\sin(r_1' y)+\frac{k_2' K'}{T_1' T_2'}y\sin(r_2' y)\right)- \tag{I6}$$

$$-\frac{k_1 K'}{T_1' T_2'}\cos(r_1' y)=0$$

$$I(y)=y^3-\frac{1}{T_1' T_2'}y-\frac{k_2' K' T_3'}{T_1' T_2'}y^2\sin(r_2' y)-\left(\frac{k_1 K' T_3'}{T_1' T_2'}y\cos(r_1' y)+\frac{k_2' K'}{T_1' T_2'}y\cos(r_2' y)\right)+ \tag{I7}$$

$$+\frac{k_1 K'}{T_1' T_2'}\sin(r_1' y)=0$$

By considering identical delays, $r_1' = r_2' = r'$, and by raising, as in section 3, to second order, we obtain finally the following cubic equation of y^2 whose solutions can be found analytically:

$$\left(y^2\right)^3 + \left(\frac{T_1'^2 + T_2'^2 - k_2' K'^2 T_3'^2}{T_1'^2 T_2'^2}\right)\left(y^2\right)^2 + \frac{1 - K'^2\left(k_1^2 T_3'^2 + k_2'\right)}{T_1'^2 T_2'^2}\left(y^2\right) - \frac{k_1^2 K'^2}{T_1'^2 T_2'^2} = 0 \qquad \text{(I8)}$$