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A note on a simplified model of ship yawing in steep following seas

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ABSTRACT: The possibility of realizing a mechanism of parametric instability concerning the horizontal plane dynamics of a ship is discussed. The background of such investigations is given first that goes back to more than fifty years. Mathematical models based of ship yawing in quartering waves are developed: one such model is based on the standard manoeuvring equations for sway and yaw. Another one is derived from the simplified Nomoto equation on the basis of a more phenomenological-type viewpoint. Requirements for the "equivalence" between these two models are identified. Necessary conditions for ensuring yaw stability that combine parameters of the ship, of the waves and of the steering control system are also produced.

1 REVISITING AN OLD PROBLEM

The line of argument behind the present note is traced back to the seminal paper of Weinblum & St Denis (1950) who had realised that the fundamental equations governing ship manoeuvring in waves feature time-dependent coefficients i.e. for harmonic waves they should be Hill's-type (in their more rudimentary form, Mathieu-type). Soon however this idea was challenged: Rydill (1959) suggested that, in a consistent linear approach, the wave slope should be treated as a small perturbation parameter, likewise heading's deviation from the intended direction of motion. Thus, at first-order approximation, the product of wave slope and heading's deviation should be eliminated. It then accrues that wave excitations need not be calculated at the time-varying instantaneous heading ψ of the ship but only at the desired heading that is a fixed control parameter. Subsequently, instead of a system featuring combined direct and parametric excitations in the directions of sway and yaw, a system that is purely directly excited is obtained.

Rydill had felt that the $Ak\,\psi$ terms produced 'unreasonable' solutions. Weinblum in his discussion of Rydill's paper was critical about the prospect of such an approach for explaining extreme phenomena of ship behaviour in steep waves, especially at low encounter frequencies. Uncomfortable with the linearization were apparently also Wahab & Swaan (1964) who, in their early theory of broaching, calculated the wave excitations for various longitudinal positions of a ship on the wave, with reference to the instantaneous heading; i.e. terms $Ak\,\psi$ were kept. Parenthetically, their approach was restrictive in a different sense: by tackling only zero-frequency-of-encounter scenarios, the dynamics owed to the overtaking waves remained out of scope. In general it is not too difficult to call upon a number of arguments supporting the inclusion of the $Ak\,\psi$ term (such arguments are collected in Appendix I).

Numerical predictions of resonant-type broaching behaviour in astern seas revived the discussions about parametric-type yawing (Spyrou 1997). Such predictions seem to corroborate experiential reports of a "cumulative" type of broaching, alleged to arise in very steep astern seas and marked by the gradual build-up of yaw amplitude in successive wave encounters. This however is reminiscent of a manifestation of yaw resonance. Such a version of broaching should represent a hazard for larger, displacement-type, vessels operating with moderate speed that is, yet, well below the wave celerity of long waves.

The introduction of a phenomenological heading-dependent wave excitation term to the simplified Nomoto equation, combined with a standard rudder control law, yield a Mathieu-type yaw equation of the instantaneous heading with the following noteworthy features:

– the parametric term represents basically the intensity of wave excitation;
– damping is governed by the differential term of rudder control;
– a bias term appears at the right-hand-side that depends on the desired heading.

The condition of stability at principal resonance of this damped Mathieu-type system is provided by a well-known relationship between the damping ratio and the parametric amplitude (e.g. Berge et al. 1984). Unfortunately though, due to the phenomenological nature of this yaw equation, it is not clear how to determine the parametric amplitude from wave pressure integrations on the hull (or from direct force measurements on "captive" models). This is the main point that is clarified in the present note.

Tackling the problem from another end, one might had opted to start from the linear pair of sway and yaw equations of standard manoeuvring theory; and thereafter to incorporate wave excitations in the sway and yaw directions, whose calculation or measurement is straight-forward. Finally decouple the system to obtain a single 3rd or higher order equation of single yaw with time-periodic coefficients. The stability condition of this expanded system is another issue that will be treated below. Firstly however we shall lay the mathematical details linked to these two points of view.

2 CALCULATION OF THE AMPLITUDE OF PARAMETRIC EXCITATION

The customary pair of linear sway, yaw equations, without memory effect, can be written, after incorporation of wave excitation terms at their right-hand-sides, indicated by subscript w, as follows:

Sway:

$$\left(m'-Y'_{\dot v}\right)\dot v' - Y'_v v' + \left(m' x'_G - Y'_{\dot r}\right)\dot r' + \\ + \left(m' - Y'_r\right) r' = Y'_\delta\,\delta + F'_w$$

Yaw:

$$\left(m' x'_G - N'_{\dot v}\right)\dot v' - N'_v v' + \left(I'_z - N'_{\dot r}\right)\dot r' + \\ + \left(m' x'_G - N'_r\right) r' = N'_\delta\,\delta + M'_w \qquad (1)$$

Primes indicate nondimensionalised quantities. Symbols are used with their customary meaning in manoeuvring theory. Wave loads in sway and yaw may be expressed respectively, for small ψ, as follows:

$$F'_w = Y'_w \psi \sin\left(\omega'_e t'\right)$$
$$M'_w = N'_w \psi \cos\left(\omega'_e t'\right) \qquad (2)$$

We use a simple rudder control law with proportional and differential gains and at this instance we disregard delay:

$$\delta = -k_1\left(\psi - \psi_r\right) - k'_2 \dot\psi' \qquad (3)$$

With substitution of (2) and (3) into (1) and after several standard transformations, it can be shown that the above lead to the following 3rd order ordinary differential equation of yaw (Spyrou 1997):

$$T'_1 T'_2 \psi'^{(3)} + \left(T'_1 + T'_2 + k_2 K' T'_3\right)\ddot\psi' + \\ + \left[1 + k_1 K' T'_3 + k_2 K' + W' \cos\left(\omega'_e t' - a\right)\right]\dot\psi' + \\ + \left[k_1 K' - R' \cos\left(\omega'_e t' - \beta\right)\right]\psi = k_1 K' \psi_r \qquad (4)$$

Immediately should be noticed the presence of time-dependent coefficients in two places. These can be calculated according to the following expressions, assuming a body-fixed system of axes placed at the ship's centre of gravity (Spyrou 1997):

$$W' = \frac{\sqrt{N'^2_{\dot v}\, Y'^2_w + \left(m' - Y'_{\dot v}\right)^2 N'^2_w}}{Y'_v N'_r - N'_v\left(Y'_r - m'\right)}$$

$$V' = \frac{\sqrt{N'^2_v\, Y'^2_w + Y'^2_v\, N'^2_w}}{Y'_v N'_r - N'_v\left(Y'_r - m'\right)}$$

$$R' = \sqrt{\omega'^2_e W'^2 + V'^2 + 2\omega'_e W' V' \cos\left(\gamma - \varepsilon\right)} \approx \\ \approx \left|V' + W'\,\omega'_e\right| \qquad (5)$$

The expressions of phase angles are:

$$\alpha = \arcsin\frac{N'_{\dot v}\, Y'_w}{W'}$$
$$\beta = \arcsin\frac{-\left(m' - Y'_{\dot v}\right) N'_w \omega'_e - N'_v\, Y'_w}{R'}$$
$$\gamma = \arcsin\frac{-\left(m' - Y'_{\dot v}\right) N'_w}{W'}$$
$$\varepsilon = \arcsin\frac{-N'_v\, Y'_w}{V'} \qquad (6)$$

Obviously, the amplitudes of the parametric terms should be functions of the wave profile. To simplify the analysis, let us make from now on the (approximately correct) assumption that the wave yaw moment is in phase with the wave while the wave sway force is at $\pi/2$. phase difference.

Suppose one had started from the simplified Nomoto equation:

$$T'\ddot\psi' + \dot\psi' = K'\delta \qquad (7)$$

A heuristic wave yaw moment should be like:

$$M'_{wave} = A'\psi \cos\left(\omega'_e t'\right) \qquad (8)$$

where A' is a phenomenological wave excitation coefficient. Combination of the above with the control law (2) produces the following alternative form of the yaw equation:

$$T'\ddot\psi' + \left(1 + k_2 K'\right)\dot\psi' + \left(k_1 K' - A' \cos\omega'_e t'\right)\psi = k_1 K' \psi_r \qquad (9)$$

In the above the amplitude A' is unknown. In order to determine this, we should deduce the relationship of

A' with the W', V', R' terms of equation (4) which are directly linked to hull geometry. To this end, all terms of (9) were divided by k_1K':

$$\frac{1}{\underbrace{\left(\frac{k_1K'}{T'}\right)}_{\omega_0^2}}\,\ddot{\psi}' + \frac{\left(1+k_2K'\right)}{k_1K'}\dot{\psi}' + \left(1-\frac{A'}{k_1K'}\cos\omega'_e t'\right)\psi = \psi_r \qquad (10)$$

The same division applied to (4) yields:

$$\frac{T_1'T_2'}{k_1K'}\dddot{\psi}' + \frac{1}{\frac{k_1K'}{\left(T_1'+T_2'+k_2K'T_3'\right)}}\ddot{\psi}' + \left[T_3' + \frac{W'}{k_1K'}\cos\left(\omega'_e t' - a\right) + \frac{1+k_2K'}{k_1K'}\right]\dot{\psi}' + \left[1-\frac{R'}{k_1K'}\cos\left(\omega'_e t' - \beta\right)\right]\psi = \psi_r \qquad (11)$$

For (9) to coincide approximately with (11), it suffices thus to satisfy the following relations:

$$\begin{aligned}
&\frac{T_1'T_2'}{k_1K'} \cong 0 \quad (\text{i.e. } T_1'T_2' << k_1K'),\\
&T_3' \cong 0,\\
&T_1'+T_2'+k_2K'T_3' \cong T'\\
&\frac{W'}{k_1K'} \cong 0,\\
&\beta \cong 0,\\
&A' = R'
\end{aligned} \qquad (12)$$

The first three conditions refer to still-water behaviour of a ship with steering control and they can be assumed as necessary for achieving correspondence between the simplified and the full Nomoto equation. The remaining set of three conditions encapsulates the wave effect. W' should be a small quantity relatively to customary values of k_1K' even for extreme waves. The same applies for the condition $\beta \cong 0$. With all other requirements (12) fulfilled, the amplitude of wave excitation A' should thus be determined simply from the relationship:

$$A' = R' \cong \left|V' + W'\omega'_e\right| \qquad (13)$$

A more elaborate investigation on this equivalence using the Laplace transform of the two models is given in Appendix II. Comparisons between the 2nd and 3rd order models in terms of the yaw angle (vertical axis) as function of nondimensional time $t(U/L)$ are shown in Figures 1 to 5 for a medium size Japanese fishing vessel. Differences arise mainly in the transient part of yaw response. The speed influences behaviour through the frequency of encounter that appears in

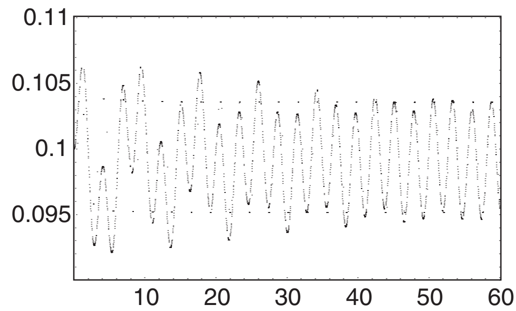

Figure 1. Comparison of responses (continuous line corresponds to 3rd order model): $U = 5$ m/s, $k_1 = 3$, $k_2' = 1$, $\psi_0 = 0.1$ rad, $\psi_r = 0.1$ rad.

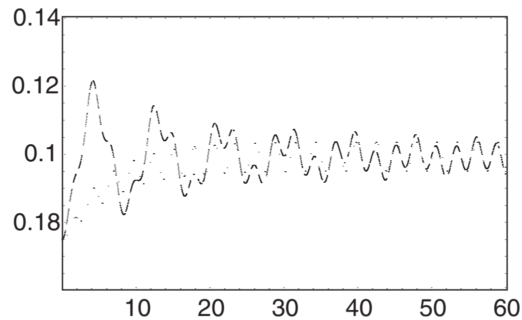

Figure 2. As above, with $k_1 = 3$, $k_2' = 1$, $\psi_0 = 0.075$ rad.

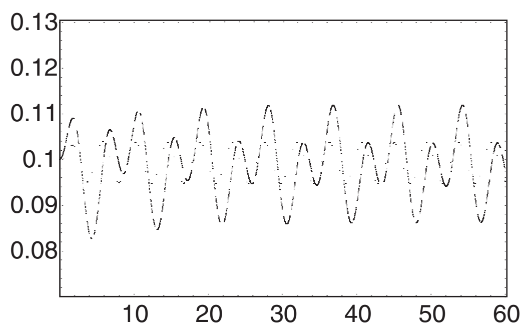

Figure 3. As Fig. 1, with $U = 7$ m/s, $k_1 = 3$.

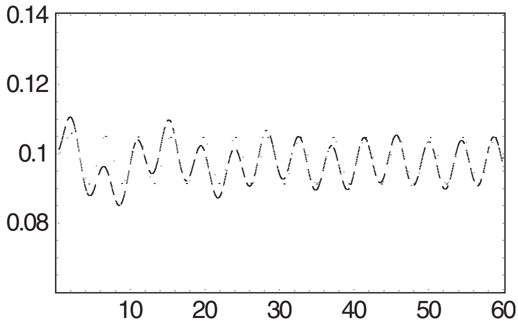

Figure 4. As above, with $U = 7$ m/s, $k_1 = 1$.

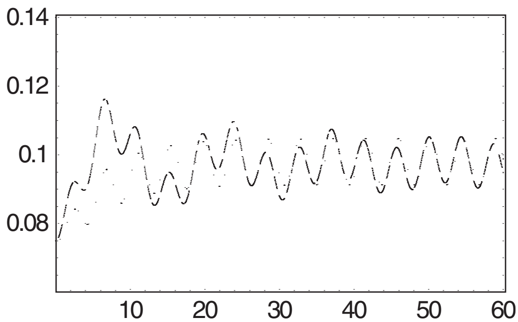

Figure 5. As above, with $\psi_0 = 0.075$ rad.

the expressions of indirect wave excitation R', and subsequently of A'.

3 CONDITIONS OF STABILITY OF "REDUCED ORDER" SYSTEM

The conditions of stability of the reduced system (9) are described below. As well-known, straight-line stability in calm-sea is guaranteed if the differential gain satisfies the following inequality (see for example Koyama 1972):

$$k_2' > -\frac{1}{K'} \tag{14}$$

The above can be deduced directly from (9), by requesting the term that plays the role of damping to be positive. In following waves, a quasi-static loss of course stability at zero frequency-of-encounter is avoided, if the amplitude of the sinusoidal stiffness term is less than one. This leads to the following condition:

$$A' < k_1 \left| K' \right| \tag{15}$$

There is also a "dynamic" condition of stability to be satisfied, concerning non-zero frequencies of encounter. This safeguards against the occurrence of parametric-type resonant instability of yaw. As is well known, for a damped, parametrically excited linear oscillator at exact principal resonance, the critical (nondimensional) amplitude of parametric excitation is 4 times the damping ratio (Berge et al. 1987). In the vicinity of principal resonance the boundary is given by the formula:

$$h_{crit} = \sqrt{\left(2 - \frac{1}{2}\frac{\omega_e'^2}{\omega_0'^2}\right)^2 + \left(2\frac{\omega_e'}{\omega_0'}\zeta\right)^2} \tag{16}$$

Strictly speaking, the above should hold true for infinitesimal damping ratio. Nonetheless it is known that (16) is still a good predictor of the critical parametric forcing as long as the damping ratio remains relatively low. For controlled yaw motion the damping ratio ζ could be determined by the following expression:

$$\zeta = \frac{1}{2}\frac{\left(\frac{1}{T'} + k_2'\frac{K'}{T'}\right)}{\sqrt{k_1\frac{K'}{T'}}} \tag{17}$$

It should be observed that the denominator $\sqrt{k_1 K' T'}$ stands for the nondimensional yaw natural frequency (it has a meaning only if there is angular feedback). Yaw's damping ratio increases linearly with k_2' and reduces with the square root of k_1. It is observed that gain values determine to a large extent whether the system's damping ratio is low or high. For moderate damping one might use the analytical formula of Gunderson et al (1974), which however provides only a crude prediction of the vertex of the instability region and is not really recommended (Leiber & Risken 1988). The rule that the minimal parametric forcing which could create instability is about 4 times the damping ratio seems to work well even for moderate damping values.

Let us assume that the selected gain values are relatively insufficient and they produce for the system a relatively low yaw damping ratio. In such a case it would be valid to determine the critical parametric forcing from the simplified expression $|h_{crit}| < 4\zeta$. Then, by combining with (17) the following dynamic stability condition is obtained:

$$A' < \left[2\frac{\left(\frac{1}{T'} + k_2'\frac{K'}{T'}\right)}{\sqrt{k_1\frac{K'}{T'}}}\right] k_1 \left|K'\right| \tag{18}$$

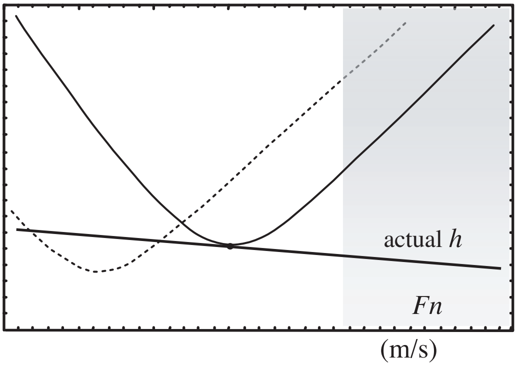

Figure 6. The condition of parametric yawing is produced by the intersection of the curves of actual and critical h. The position of these curves depends on proportional gain; however that of h_{cr} is very sensitive to the differential.

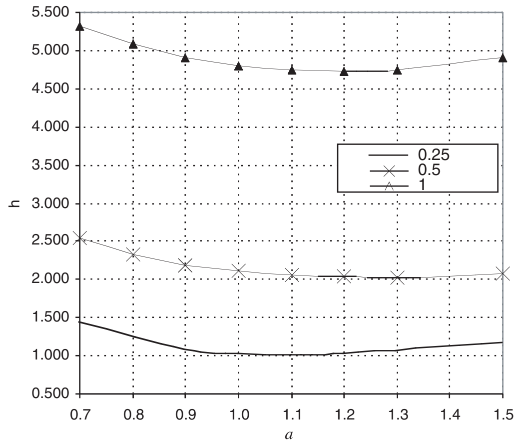

Figure 7. Numerical boundary of parametric yawing for large ζ. By a is meant the ratio $4\omega_0^2/\omega_e^2$.

The above "dynamic" condition (18) may be contrasted against the semi-static condition (15). The term in brackets at the right-hand-side of (19), specifically if it is larger or smaller than 1, determines whether dynamic instability of yaw is incurred at higher or lower waves compared to the semi-static scenario. In other words, in a slightly damped system (which grossly means a small k_2) condition (1) is more stringent that (15) while the opposite is likely to happen at high damping. It is recalled that the A' forcing term depends, according to expression (13), on the frequency of encounter and thus it depends also on the Froude number where the ship operates at some instant.

Parametric oscillations will arise if the critical level of parametric excitation, provided for example by expression (16) of h_{cr} locus, is reached by the actual parametric excitation $h = A'/(k_1 K')$. This is depicted qualitatively in Fig. 6. If the curves of critical and actual h intersect, parametric oscillations should be expected. It is notable that the proportional gain influences both curves. On the other hand, it is evident that the distance of the two curves is very sensitive to the differential gain. A small reduction of k_2 bears a disproportionally large effect on h_{cr}. Furthermore, strictly speaking, for a large k_2 (where as a result the damping ratio has become large), the layout of the curve itself should change and expression (16) may no longer be considered as valid. The (numerical) stability diagram for arbitrary damping has been discussed in Leiber & Risken (1988). In Figure 7 are collected some characteristic boundaries that we have produced numerically, spanning a wide range of damping ratios, for unit natural frequency.

It is known that, the phenomenon of surge dynamics known as surf-riding may occur at higher Froude numbers. This region is indicated in Fig. 6 as grey. The overlap of regions where different phenomena of instability concerning different directions of ship motions could take place should be analysed before conclusions are drawn for the susceptibility towards some kind of instability.

4 CONCLUDING REMARKS

Whilst the mechanism of instability described above is established theoretically, it is still unknown what ship types and sizes could be prone for exhibiting it. It is likely however that it is relevant for ships of larger size. Use of the wave excitation term A' [equation (13)] should be done with caution, as the condition of equivalence on which it is based could hold only for certain ships. Therefore, further research will be required on this. However one thing that may be said is that, in comparison to a scenario of semi-static loss of stability on the horizontal plane which would correspond to the classical type of broaching, larger waves will be required for this mechanism to be realised, due to the large damping in yaw that normally exists. This could be reversed however in case that very sloppy rudder control is applied on the ship.

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APPENDIX I: SOME ARGUMENTS IN FAVOUR OF RETAINING THE $Ak\,\psi$ TERM

- Wave slope Ak is an independent parameter of the problem while ψ is the dependent variable. The amplitude of ψ should depend on the encountered waves (i.e. Ak and the wave length to ship length ratio λ/L), the method of steering and the intrinsic hull and rudder characteristics. It is possible to have small ψ for extreme Ak and vice versa.
- An upper limit of Ak is about $\pi H/\lambda = \pi/7 \approx 0.45$ which is not a truly small quantity. It is notable that, a steep wave with, say, $\lambda/L = 1, H/\lambda = 1/10$, brings about an effect on sectional draught (assuming equilibrium in the vertical direction) up to $\pm L/20$ which is of the order of ship draught in calm sea. Therefore, even if the wave slope were taken as a small quantity, the effect of wave profile on the submerged part of the hull, which determines wave excitation, can be quite significant.
- It is known that the wave yaw moment could change significantly if the disturbance of the incident wave for very low encounter frequency were included in the calculations. According to Okhusu's (1986) calculation method, a two-fold increase of amplitude, compared to the corresponding Froude-Krylov value, may not be uncommon if all terms up to the lowest order of magnitude are taken into account, even if these are of higher order compared to Froude-Krylov.
- Had one neglected terms with products of angle and wave slope, parametric instability in roll due to the wave contouring effect should have become impossible too; which we know to be incorrect.

APPENDIX II: EQUIVANENCE OF MODELS THROUGH THE LAPLACE TRANSFORM

Series expansion of the $\cos(\omega_e' t')$ term of (1) gives:

$$cos(\omega_e' t')\psi(t') = 1 - \frac{\omega_e'^2}{2!}t'^2\psi(t') + \frac{\omega_e'^4}{4!}t'^4\psi(t') - \cdots \tag{AII1}$$

We shall use the following well-known identity for the derivative of the image function:

$$\text{If} \quad L[f(t)] = F(s) \ \text{ then } \ L\left[(-1)^n t^n f(t)\right] = F^{(n)}(s) \tag{AII2}$$

The Laplace transform of (5), after substitution of (AII1) and use of (AII2), yields (we assumed that $\psi(0) = \psi'(0) = 0$):

$$T s^2 \Psi(s) + (1 + k_2 K) s \Psi(s) + k_1 K \Psi(s) - \\ -A\left[\Psi(s) - \frac{\omega_e^2}{2!}\Psi^{(2)}(s) + \frac{\omega_e^4}{4!}\Psi^{(4)}(s) - \cdots\right] = \frac{k_1 K \psi_r}{s} \tag{AII3}$$

The above is recast as:

$$\left[T s^2 + (1 + k_2 K)s + k_1 K - A\right]\Psi(s) + \\ +A\frac{\omega_e^2}{2!}\Psi^{(2)}(s) - A\frac{\omega_e^4}{4!}\Psi^{(4)}(s) - \cdots = \frac{k_1 K \psi_r}{s} \tag{AII4}$$

If we request ω_e raised to 2 or higher to be approximately equal to zero the above becomes:

$$\left[T s^2 + (1 + k_2 K)s + k_1 K - A\right]\Psi(s) = \frac{k_1 K \psi_r}{s} \tag{AII5}$$

Similarly, we apply the Laplace transform for the parts of equation (1), with the same assumptions:

$$L\left[T_1' T_2' \psi^{(3)}(t)\right] = T_1' T_2' \, s^3 \, \Psi(s) \tag{AII6}$$

$$L\left[(T_1' + T_2' + k_2 K' T_3')\psi^{(2)}(t)\right] = (T_1' + T_2' + k_2 K' T_3') s^2 \, \Psi(s) \tag{AII7}$$

$$L\left[(1 + k_1 K' T_3' + k_2 K')\dot{\psi}(t)\right] = (1 + k_1 K' T_3' + k_2 K') s \, \Psi(s) \tag{AII8}$$

$$L\left[W' \cos(\omega_e t - \sigma_1)\dot{\psi}'(t)\right] = \\ W' L\left[\cos\sigma_1 \dot{\psi}'(t') + \omega_e t \sin\sigma_1 \dot{\psi}'(t) - \frac{\omega_e^2 t^2 \cos\sigma_1}{2!}\dot{\psi}'(t) - \right. \\ \left. - \frac{\omega_e^3 t^3 \sin\sigma_1}{3!}\dot{\psi}'(t) + \frac{\omega_e^4 t^4 \cos\sigma_1}{4!}\dot{\psi}'(t') - \cdots\right] \tag{AII9}$$

After some manipulation (AIII9) becomes:

$$L\left[W'\cos\left(\omega_e t-\sigma_1\right)\dot\psi'\left(t'\right)\right]=W'\left(s\cos\sigma_1-\omega_e\sin\sigma_1\right)\Psi\left(s\right)-$$
$$-\frac{W\omega_e}{1!}\left(s\sin\sigma_1-\omega_e\cos\sigma_1\right)\Psi'\left(s\right)+$$
$$+\frac{W\omega_e^2}{2!}\left(s\cos\sigma_1-\omega_e\sin\sigma_1\right)\Psi^{(2)}\left(s\right)+$$
$$+\frac{W'\omega_e^3}{3!}\left(s\sin\sigma_1+\omega_e\cos\sigma_1\right)\Psi^{(3)}\left(s\right)+\frac{W'\omega_e^4}{4!}\cos\sigma_1\Psi^{(4)}\left(s\right)+\cdots \tag{AII10}$$

To derive (AII10) we used the identity:

$$L\left[t^n\,\dot\psi\left(t\right)\right]=-\left[s\,\Psi\left(s\right)\right]^{n)} \tag{AII11}$$

This relation is produced as follows:
The Laplace transform of $t\,\dot\psi\,(t)$ is:

$$L\left[t\dot\psi\left(t\right)\right]=\int_0^{+\infty}e^{-st}\left[t\dot\psi\left(t\right)\right]dt=\int_0^{+\infty}t\,e^{-st}\frac{d\psi\left(t\right)}{dt}dt=$$
$$=\lim_{a\to+\infty}\left[a\,e^{-sa}\,\psi\left(a\right)-0-\int_0^a\frac{d\left(t\,e^{-st}\right)}{dt}\psi\left(t\right)dt\right]$$

For a conventional (say sinusoidal–type) yaw response $\psi(t)$, the term $\lim_{a\to+\infty} ae^{-sa}\,\psi(a)$ becomes zero for $s>0$. Therefore,

$$L\left[t\dot\psi\left(t\right)\right]=-\lim_{a\to+\infty}\left[\int_0^a e^{-st}\psi\left(t\right)dt-s\int_0^a t\,e^{-st}\psi\left(t\right)dt\right]=$$
$$=-\Psi\left(s\right)-s\,\Psi'\left(s\right)=-\left[s\,\Psi\left(s\right)\right]'$$

The above is easily extended to the general expression (AII11). Also:

$$L\left[R'\cos\left(\omega_e t-\rho\right)\dot\psi\left(t\right)\right]=R'\cos\rho\,\Psi\left(s\right)-$$
$$-R'\omega_e\sin\rho\,\Psi'\left(s\right)-\frac{R'\omega_e^2\cos\rho}{2!}\Psi^{(2)}\left(s\right)+$$
$$+\frac{R'\omega_e^3\sin\rho}{3!}\Psi^{(3)}\left(s\right)+\frac{R'\omega_e^4\cos\rho}{4!}\Psi^{(4)}\left(s\right)+\cdots \tag{AII12}$$

$$L\left[k_1K'\psi\left(t\right)\right]=k_1K'\,\Psi\left(s\right) \tag{AII13}$$

$$L\left[k_1K'\psi_r\right]=\frac{k_1K'\psi_r}{s} \tag{AII14}$$

The Laplace transformed version of equation (1) is then obtained:

$$\left[T_1'\,T_2'\,s^3+\left(T_1'+T_2'+k_2'K'T_3'\right)s^2+\left(1+k_1K'T_3'+k_2'K'+W'\cos\sigma_1\right)s+\right.$$
$$\left.+\left(k_1K'-R'\cos\rho-W'\,\omega_e'\sin\sigma_1\right)\right]\Psi\left(s\right)-$$
$$-\left[W'\,\omega_e'\sin\sigma_1\;s+\left(W'\,\omega_e'^2\cos\sigma_1+R'\,\omega_e'\sin\rho\right)\right]\Psi^{(1)}\left(s\right)$$
$$+\left[\frac{W'\,\omega_e'^2}{2!}\cos\sigma_1\,s-\left(\frac{W'\,\omega_e'^3}{2!}\sin\sigma_1+\frac{R'\,\omega_e'^2}{2!}\cos\rho\right)\right]\Psi^{(2)}\left(s\right)+$$
$$+\left[\frac{W'\,\omega_e'^3}{3!}\sin\sigma_1\,s+\left(\frac{W'\,\omega_e'^4}{3!}\cos\sigma_1+\frac{R'\,\omega_e^3}{3!}\sin\rho\right)\right]\Psi^{(3)}\left(s\right)+\cdots=$$
$$=\frac{k_1K'\psi_r}{s} \tag{AII15}$$

By setting ω_e^2 or higher equal to zero the above becomes:

$$\left[T_1'T_2's^3+\left(T_1'+T_2'+k_2'K'T_3'\right)s^2+\left(1+k_1K'T_3'+k_2'K'+W'\cos\sigma_1\right)s\right.$$
$$\left.+\left(k_1K'-R'\cos\rho-W'\,\omega_e'\sin\sigma_1\right)\right]\Psi\left(s\right)-$$
$$-\left(W'\,\omega_e'\sin\sigma_1\;s+R'\,\omega_e'\sin\rho\right)\Psi^{(1)}\left(s\right)=\frac{k_1K'\psi_r}{s} \tag{AII16}$$

In general, the angles σ_1, ρ are small; hence products of their sine with ω_e could be neglected. For consistence we assume that $\cos\sigma_1\approx\cos\,\rho\approx 0$. Then (AII16) becomes:

$$\left[T_1'T_2's^3+\left(T_1'+T_2'+k_2'K'T_3'\right)s^2+\left(1+k_1K'T_3'+k_2'K'+W'\right)s+\right.$$
$$\left.+\left(k_1K'-R'\right)\right]\Psi\left(s\right)=\frac{k_1K'\psi_r}{s} \tag{AII17}$$

Therefore:

$$\frac{\Psi\left(s\right)}{\psi_r}=$$
$$=\frac{k_1K'}{s\left[T_1'\,T_2'\,s^3+\left(T_1'+T_2'+k_2K'T_3'\right)s^2+\left(1+k_1K'T_3'+k_2'K'+W'\right)s+\left(k_1K'-R'\right)\right]} \tag{AII18}$$

Series expansion of the above yields that the right-hand-side is approximately equal to:

$$\frac{\Psi\left(s\right)}{\psi_r}\approx\frac{k_1K'}{\left(k_1K'-R'\right)s}-\frac{k_1K'\left(1+k_2'K'+k_1K'T_3'+W'\right)}{\left(k_1K'-R'\right)^2}+$$
$$+\frac{k_1K'\left[\left(1+k_2'K'+k_1K'T_3'+W'\right)^2-\left(T_1'+T_2'-k_2'K'T_3'\right)\left(k_1K'-R'\right)\right]s}{\left(k_1K'-R'\right)^3}+$$
$$+O\left[s\right]^2 \tag{AII19}$$

Similarly for the simplified yaw equation (4):

$$\frac{\Psi(s)}{\psi_r} = \frac{k_1 K'}{s\left[T's^2 + (1+k_2'K')s + (k_1K' - A')\right]} \approx$$

$$\approx \frac{k_1K'}{(k_1K'-A')s} - \frac{k_1K'(1+k_2'K')}{(k_1K'-A')^2} + \tag{AII20}$$

$$+\frac{k_1K'\left[(1+k_2'K')^2 - T'(k_1K'-A')\right]s}{(k_1K'-A')^3} + O[s^2]$$

To obtain identical leading-order terms the condition thus is:

$$A' = R' \approx V' + W'\cos(\sigma_2 - \mu)\omega_e' \approx V' + W'\omega_e' \tag{AII21}$$

APPENDIX III: CONDITION OF INSTABILITY FOR THE 3RD-ORDER MANOEUVRING MODEL

One assumes the possibility the 3rd-rder system to exhibit a mechanism of instability manifested by a growing oscillatory yaw motion when the period of encounter is about half the natural period in yaw. Let us scale time by setting $\omega_e t = 2\tau$. Then

$$\frac{d\psi}{dt} = \frac{\omega_e}{2}\frac{d\psi}{d\tau},\quad \frac{d^2\psi}{dt^2} = \frac{\omega_e^2}{4}\frac{d^2\psi}{d\tau^2},\quad \frac{d^3\psi}{dt^3} = \frac{\omega_e^3}{8}\frac{d^3\psi}{d\tau^3}$$

Substitution into (1) yields the following yaw equation:

$$\frac{d^3\psi}{d\tau^3} + \underbrace{\frac{2}{\omega_e'}\left(\frac{1}{T_1'} + \frac{1}{T_2'} + \frac{k_2K'T_3'}{T_1'T_2'}\right)}_{a_1}\frac{d^2\psi'}{d\tau^2} +$$

$$+\left[\underbrace{\frac{4}{\omega_e'^2}\left(\frac{1}{T_1'T_2'} + \frac{k_1K'T_3'}{T_1'T_2'} + \frac{k_2K'}{T_1'T_2'}\right)}_{a_2} + \underbrace{\frac{4}{\omega_e'^2}\frac{W'}{T_1'T_2'}}_{a_3}\cos(2\tau - \sigma_1)\right]\frac{d\psi'}{d\tau} +$$

$$+\left[\underbrace{\frac{8}{\omega_e'^3}\frac{k_1K'}{T_1'T_2'}}_{a_4} - \underbrace{\frac{8}{\omega_e'^3}\frac{R'}{T_1'T_2'}}_{a_5}\cos(2\tau-\rho)\right]\psi = \underbrace{\frac{8}{\omega_e'^3}\frac{k_1K'}{T_1'T_2'}}_{b}\psi_r \tag{AIII 1}$$

At this stage we omit the bias term at the right hand side, i.e. $b = 0$ (which means also that $\psi_r = 0$). The solution could be assumed to have the form $\psi \approx \psi_{01}e^{\sigma\tau} + \psi_{02}e^{\mu\tau}\cos(\tau + \vartheta)$. Let us seek the stability condition for the special case that, in the eigendirection associated with eigenvalue σ there is strong contraction i.e. the system could show instability only in an oscillatory mode as the real part μ of the complex pair of eigenvalues turns positive. In other words we shall assume that after short time the solution almost coincides with the expression $\psi \approx \psi_{02}e^{\mu\tau}\cos(\tau + \vartheta)$ which is thus the expression to be substituted in (10). Then, after some manipulation, one obtains:

$$e^{\mu\tau}\left[\mu^3\cos(\tau+\vartheta) + \sin(\tau+\vartheta)\right] +$$

$$+a_1 e^{\mu\tau}\left[\mu^2\cos(\tau+\vartheta) - \cos(\tau+\vartheta)\right] +$$

$$+\left[a_2 + a_3\cos(2\tau-\sigma_1)\right]e^{\mu\tau}\left[\mu\cos(\tau+\vartheta) - \sin(\tau+\vartheta)\right] +$$

$$+\left[a_4 + a_5\cos(2\tau-\rho)\right]e^{\mu\tau}\cos(\tau+\vartheta) = 0 \tag{AIII 2}$$

Separating $\sin\tau$ from $\cos\tau$ terms and neglecting sines and cosines of 3τ leads to having to satisfy the equation:

$$\left[(\mu^3 + a_1\mu^2 - a_1 + a_2\mu + a_4)\cos\vartheta + \frac{a_3\mu}{2}\cos(\sigma_1+\vartheta) + \right.$$

$$\left. + \frac{a_5}{2}\cos(\vartheta+\rho) + (1-a_2)\sin\vartheta - \frac{a_3}{2}\sin(\sigma_1+\vartheta)\right]\cos\tau +$$

$$+\left[-(\mu^3 + a_1\mu^2 - a_1 + a_2\mu + a_4)\sin\vartheta + \frac{a_3\mu}{2}\sin(\sigma_1+\vartheta) + \right.$$

$$\left. + \frac{a_5}{2}\sin(\vartheta+\rho) + (1-a_2)\cos\vartheta + \frac{a_3}{2}\cos(\sigma_1+\vartheta)\right]\sin\tau = 0 \tag{AIII 3}$$

For the left-hand-side to be equal to zero for all τ, the coefficients of the sine and cosine terms should be zero. These two conditions lead to the following pair of equations in terms of $\sin\vartheta$, $\cos\vartheta$:

$$\left[(\mu^3 + a_1\mu^2 - a_1 + a_2\mu + a_4) + \frac{a_3\mu}{2}\cos\sigma_1 + \right.$$

$$\left. + \frac{a_5}{2}\cos\rho - \frac{a_3}{2}\sin\sigma_1\right]\cos\vartheta +$$

$$+\left[-\frac{a_3\mu}{2}\sin\sigma_1 - \frac{a_5}{2}\sin\rho + (1-a_2) - \frac{a_3}{2}\cos\sigma_1\right]\sin\vartheta = 0 \tag{AIII 4}$$

$$-(\mu^3 + a_1\mu^2 - a_1 + a_2\mu + a_4) + \frac{a_3\mu}{2}\cos\sigma_1 + $$

$$\left. + \frac{a_5}{2}\cos\rho - \frac{a_3}{2}\sin\sigma_1\right]\sin\vartheta +$$

$$+\left[\frac{a_3\mu}{2}\sin\sigma_1 + \frac{a_5}{2}\sin\rho + (1-a_2) + \frac{a_3}{2}\cos\sigma_1\right]\cos\vartheta = 0 \tag{AIII 5}$$

The system of homogeneous equations with unknowns $\sin\vartheta$ and $\cos\vartheta$ receives a solution only if the determinant is zero, i.e.:

$$\left(\frac{a_3\mu}{2}\cos\sigma_1+\frac{a_5}{2}\cos\rho-\frac{a_3}{2}\sin\sigma_1\right)^2-$$
$$-\left(\mu^3+a_1\mu^2-a_1+a_2\mu+a_4\right)^2= \qquad \text{(AIII 6)}$$
$$=\left(1-a_2\right)^2-\left(-\frac{a_3\mu}{2}\sin\sigma_1-\frac{a_5}{2}\sin\rho-\frac{a_3}{2}\cos\sigma_1\right)^2$$

The condition of marginal stability arises when $\mu = 0$:

$$\frac{a_3^2+a_5^2}{4}-\frac{a_3\,a_5}{2}sin(\sigma_1-\rho)=(a_4-a_1)^2+(1-a_2)^2 \qquad \text{(AIII 7)}$$

Going back to the original manoeuvring parameters we obtain:

$$4\omega_e'^2W'^2+16R'^2+16\omega_e'\sin(\sigma_1-\rho)W'R'=$$
$$=4\left[4k_1K'-\left(T_1'+T_2'+k_2K'T_3'\right)\omega_e^2\right]^2+$$
$$+\left[T_1'T_2'\omega_e'^3-4\left(1+k_1K'T_3'+k_2'K'\right)\omega_e'\right]^2 \qquad \text{(AIII 8)}$$

Given that the angles $\sigma_1, \sigma_2, \rho, \mu$ are small, the following condition of stability threshold is also approximately valid:

$$20\omega_e'^2W'^2+V^2+2\omega_e'W'V'=$$
$$4\left[4k_1K'-\left(T_1'+T_2'+k_2'K'T_3'\right)\omega_e'^2\right]^2+$$
$$+\left[T_1'T_2'\omega_e'^3-4\left(1+k_1K'T_3'+k_2'K'\right)\omega_e'\right]^2 \qquad \text{(AIII 9)}$$